

MAT120 Supplementary Lecture Notes

Overview of Numbers

- *Integers*: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- *Rational Numbers*: $\frac{1}{2}, -\frac{2}{3}$, etc., quotients of integers $\frac{a}{b}$, $b \neq 0$; finite or repeating decimal expansions.
- *Irrational Numbers*: $\sqrt{2}, 2^{\frac{1}{12}}, e, \pi$, non-finite, non-repeating decimal expansions.
- *Real Numbers*: All rational and irrational numbers.
- *Algebraic Numbers*: All real numbers which can be expressed as a root of a polynomial with rational coefficients. For example: $\sqrt{2}$ is a root (or zero) of $x^2 - 2$, since $(\sqrt{2})^2 - 2 = 0$.
- *Transcendental Numbers*: All real numbers which are not algebraic. e and π are transcendental, as well as irrational.
- *Complex Numbers*: $a + bi$, where i is an ‘imaginary’ square root of -1 , and a and b are real numbers. Complex numbers can be represented as points in the ‘complex plane’, where a is the real (or x) coordinate and b is the imaginary (or y) coordinate. The axes are called the real and imaginary axes.

It is not obvious why a number such as $\sqrt{2}$ is irrational. Indeed, your calculator does not know this fact. It can represent $\sqrt{2}$ to about 10 decimal places, as 1.414213562 but this gives no indication as to whether the sequence eventually repeats or not. To prove that this sequence never repeats itself after some point requires another approach. We can show that $\sqrt{2}$ is not a rational number by supposing that it is, and then arriving at a contradiction.

Theorem 0.1. $\sqrt{2}$ is an irrational number.

Proof: Suppose, by way of contradiction, that $\sqrt{2}$ is a rational number. Then we can write $\sqrt{2} = \frac{a}{b}$, and we can assume that this fraction is in lowest terms, which means that a and b have no common factor. We will now show that this leads to a contradiction.

First, we perform some basic algebra: $\sqrt{2} = \frac{a}{b}$ is equivalent to $(\sqrt{2})^2 = (\frac{a}{b})^2$, or $2 = \frac{a^2}{b^2}$, or $2b^2 = a^2$. Then, since a^2 factors as $2b^2$, we know that 2 must be a factor of a . This is because all prime factors of a perfect square must occur to even exponents. (For instance, the number $36 = 6^2$ has prime factorization $2^2 \cdot 3^2$.) Now, since we know that 2 is a factor of a , we can also say that 2^2 is a factor of a^2 . This gives us the new equations: $a = 2c$, and $a^2 = (2c)^2 = 4c^2$. But then we also have: $2b^2 = a^2 = 4c^2$, and cancelling a 2 on each side yields: $b^2 = 2c^2$. But now we are in the same position that before led us to conclude that 2 was a factor of a , but this time we can conclude that 2 is a factor of b . This says that a and b have the common factor 2, which is the contradiction we were looking for. Since all we have done is algebra, this must mean that our original assumption, that $\sqrt{2}$ is a *rational* number, must be false. So the only alternative is that $\sqrt{2}$ is an *irrational* number.

Frequency Ratios and Intervals

Definition 0.2. The term *frequency* refers to a measure of cycles per unit time. The most basic unit of frequency is cycles per second, or Hertz, abbreviated Hz.

Example 0.3. To walk at a frequency of approximately 1 Hz, I would need to see that my two footsteps are occurring roughly once per second.

Example 0.4. A sinusoidal graph can be used to represent oscillations of air pressure which for given frequencies are perceived as pitch. (More on sinusoidal graphs when we continue with the overview of functions!)

Geometric Series and Fractional Representation of Repeating Decimals

A finite geometric series is a sum of the type:

$$S = 1 + r + r^2 + r^3 + \cdots + r^n.$$

We can find a closed formula for S by first multiplying both sides by r :

$$r \cdot S = r \cdot (1 + r + r^2 + r^3 + \cdots + r^n) = r + r^2 + r^3 + \cdots + r^n + r^{n+1}.$$

Then we subtract to find a formula for S :

$$S - rS = (1 + r + r^2 + r^3 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^n + r^{n+1}) = 1 - r^{n+1}.$$

So we have

$$S(1 - r) = 1 - r^{n+1},$$

which yields:

$$S = 1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

The sum of an infinite geometric series: for any number r with $|r| < 1$:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.$$

This can be used to find the fractional representation of any repeating decimal expansion.

Example 0.5.

$$12.3434\overline{34} = 12 + \frac{34}{100} + \frac{34}{10000} + \frac{34}{1000000} + \dots \quad (1)$$

$$= 12 + \frac{34}{100} \left(1 + \frac{1}{100} + \frac{1}{10000} + \dots \right) \quad (2)$$

$$= 12 + \frac{34}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} \dots \right) \quad (3)$$

$$= 12 + \frac{34}{100} \left(\frac{1}{1 - \frac{1}{100}} \right) \quad (4)$$

$$= 12 + \frac{34}{100} \left(\frac{1}{\frac{99}{100}} \right) \quad (5)$$

$$= 12 + \frac{34}{100} \left(\frac{100}{99} \right) \quad (6)$$

$$= 12 + \frac{34}{99} \quad (7)$$

$$= \frac{12 \cdot 99 + 34}{99} \quad (8)$$

$$= \frac{1222}{99} \quad (9)$$

Frequency Ratios and Intervals

Definition 0.6. The term *frequency* refers to a measure of cycles per unit time. The most basic unit of frequency is cycles per second, or Hertz, abbreviated Hz.

Example 0.7. To walk at a frequency of approximately 1 Hz, I would need to see that my two footsteps are occurring roughly once per second.

Example 0.8. A sinusoidal graph can be used to represent oscillations of air pressure which for given frequencies are perceived as pitch. (More on sinusoidal graphs when we continue with the overview of functions!)

Definition 0.9. A *frequency ratio* between two frequencies f_1 and f_2 is simply the number f_2/f_1 . Typically $f_1 < f_2$, so $f_2/f_1 \geq 1$.

Example 0.10. Frequency ratios are related to musical intervals. For instance, what we call an *octave* corresponds to a frequency ratio of 2. For example if $f_1 = 220$ Hz and $f_2 = 440$ Hz, then $440/220$ gives a frequency ratio of 2.

Example 0.11. If two frequencies f_1 and f_2 are equal, then their ratio is 1. As a musical interval this is called a *unison*. The next simplest rational number to use as a frequency ratio is the number $3/2$. The resulting interval between two pitches with frequencies f_1 and f_2 , and ratio $f_2/f_1 = 3/2$ is called a *just perfect fifth*.

Equal Tempered Intervals and Frequency Ratios:

On a modern piano, the *semitone* or interval between two adjacent notes, has the frequency ratio:

$$2^{1/12} = 1.059463094359295264561825295 \dots$$

This is an irrational number, so the above decimal expansion goes on without ever repeating.

Other intervals on the piano have frequency ratios which are given by powers of $2^{1/12}$. For instance, a *major second* or *whole step*, which is two semitones, or the interval between two notes on the piano which are separated by one other key, is given by the frequency ratio:

$$2^{2/12} = 2^{1/6}.$$

The notes on the piano have frequencies which can be written in the form

$$440 \cdot (2^{1/12})^n$$

for various powers of n . A440 has value $n = 0$. The $A\sharp$ above A440 has value $n = 1$. The A flat below A440 has value $n = -1$. The C below A440, which is middle C , or $C4$, has values $n = -9$. This is derived by simply counting the semitones down from $A4$ to $C4$ and seeing that there are 9 steps. This means that the frequency ratio from $C4$ to $A4$ is:

$$\frac{440}{440 \cdot (2^{1/12})^{-9}} = (2^{1/12})^9.$$

Geometric Sequences and Frequency Sequences

A geometric sequence is a list of numbers (separated by commas) such that there is a common ratio between successive terms. For instance, if the entries are listed with subscripts:

$$\dots, a_i, a_{i+1}, a_{i+2}, \dots$$

then we will have

$$\frac{a_{i+1}}{a_i} = r = \frac{a_{i+2}}{a_{i+1}}$$

for some number r , called the common ratio.

A geometric sequence may be finite or infinite. Some examples of finite geometric sequences:

$$1, 3, 9, 27, 81$$

with $r = 3$, or

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}$$

with $r = \frac{1}{2}$.

Some examples of infinite geometric sequences:

$$1, 3, 9, 27, 81, 243, \dots$$

with $r = 3$, extending infinitely to the right, or

$$\dots 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$$

with $r = \frac{1}{2}$, extending infinitely to the left, or

$$\dots, \frac{1}{5^3}, \frac{1}{5^2}, \frac{1}{5}, 1, 5, 5^2, 5^3, \dots$$

with $r = \frac{1}{5}$, extending infinitely to the left and the right.

The set of frequencies on the piano form a geometric sequence which could be written:

$$440(2^{1/12})^{-48}, 440(2^{1/12})^{-47}, 440(2^{1/12})^{-46}, \dots, 440(2^{1/12})^0, 440(2^{1/12})^1, \dots, 440(2^{1/12})^{39},$$

so that there are 48 notes below $A440$ and 39 notes above $A440$. (On the modern piano there are 88 notes.)

The set of frequencies on the piano can also be given in a similar way with any starting frequency. For example, let's use the starting frequency for middle C , or $C4$:

$$440(2^{1/12})^{-9} = 261.626\dots$$

The sequence of piano frequencies can then be expressed as:

$$261.626(2^{1/12})^{-39}, 261.626(2^{1/12})^{-38}, 261.626(2^{1/12})^{-37}, \dots, 261.626(2^{1/12})^0, 261.626(2^{1/12})^1, \dots, 261.626(2^{1/12})^{48},$$

so that there are 39 notes below middle C and 48 notes above middle C .

Introduction to Pd:

First steps:

- Be careful to turn audio volume down before starting with Pd. The system has no protection built in and you can easily ruin a speaker or damage your ears with it.
- In the Pd window, under preferences, you need to set the output to use mbox pro.
- Use edit mode and run mode, toggling with the <command>-e keys. On the Mac, edit mode appears with the hand symbol, and run mode appears with the arrow.
- Put an object box onto the canvas with <command>-1.
- Type into the box "osc~ 660".
- Put another object box below and type into it "* 0.5", and connect the lower left corner of the previous osc with a cable to the top left of this new box. This takes the signal from osc and cuts the amplitude by half, for safety.

- Put another object box below and type into it “dac~”, and connect the previous box to this one with cable lines.
- In the main Pd window check the box Audio ON. You should hear a pure sine tone at 660 Hz.
- Toggle Audio ON and OFF with the control keys <command>-/ for ON and <command>- for OFF.

Other just intervals and small ratios

The just frequency ratios that we have seen so far are 1 (unison), 2 (octave), and 3/2 (just perfect fifth). Other ratios between 1 and 2, with small integer numerator and denominator are: 5/4, 4/3, 6/5, 9/8, 5/3, 16/15, 8/5 (using only prime factors 2, 3, and 5), and 7/4, 7/5, 7/6, 8/7, 9/7, etc.

Harmonic series

A vibrating string oscillates with a certain fundamental frequency determined by the length of the string and its material composition. If this frequency is F , then the string also has other natural frequencies of oscillation given by $2F$, $3F$, $4F$, $5F$, etc. All of these frequencies (up to some reasonable limit) are present in the sound of a plucked string, with various amplitudes. The loudness of these overtones, or harmonics, gives the sound its timbre, or quality. The just frequency ratios above can be obtained as ratios between these harmonic frequencies.

More on Frequency Ratios and Intervals

We discussed problem 4 from homework 2 on frequency ratios. It is important to simplify the frequency ratio to a final decimal form with the required number of decimal places accuracy. We do represent frequency ratios in other forms as well, such as:

$$2^{1/12}, 3/2, \sqrt{2},$$

but other than these special forms, a decimal value is preferred.

Envelopes

An envelope is typically a piecewise linear function. One of the common types of envelope is called ADSR which stands for Attack-Decay-Sustain-Release.

Here is an example of such a piecewise linear function:

$$f(t) = \begin{cases} 10t, & 0 \leq t < 1 \\ -3t + 13, & 1 \leq t < 3 \\ 4, & 3 \leq t \leq 13 \\ -\frac{1}{3}t + 9, & 15 \leq t \leq 27 \end{cases}$$

Note: This function is *continuous*, which means that the pieces join together without a break. The points which are connected together with line segments are:

$$(0, 0), (1, 10), (3, 4), (15, 4), \text{ and } (27, 0).$$

We can use such an envelope to mimic the sound of a plucked string or other instrument. This can be accomplished by multiplying the envelope function with a sound or signal, such as a pure sine tone.

Logarithms and cent measurement of intervals

A logarithmic function $f(x) = \log_b x$ is the function which satisfies the condition: $y = \log_b x$ if and only if $b^y = x$.

For example, the exponential equation $2^3 = 8$ can be converted to $\log_2 8 = 3$.

In other words, a logarithm is asking for an exponent, so that $\log_2 8$ is asking for the number to which I must raise the base 2 in order to get 8.

To define the cent measure of intervals or frequency ratios, we start with a frequency ratio, say F_2/F_1 , and ask for a power of 2 which gives this number. However, we also want to break up the power of two into small pieces, such that 100 pieces correspond to one semitone, or 1200 pieces fit into one octave. So we get:

$$2^{\frac{x}{1200}} = \frac{F_2}{F_1}.$$

We now define x in the above equation to be the cent measurement of the interval which has frequency ratio F_2/F_1 .

To solve for x we use logarithms. The above equation is equivalent to

$$\frac{x}{1200} = \log_2 \left(\frac{F_2}{F_1} \right).$$

Then we simply solve for x and get:

$$x = 1200 \log_2 \left(\frac{F_2}{F_1} \right).$$

Unfortunately, we usually don't have a way to compute \log_2 on our calculator, or computer. So we need to translate to other bases, like 10 (common logarithm) or e (natural logarithm). The natural logarithm is also written \ln for \log_e , where e is an irrational number which is approximately:

$$e = 2.718281828459045235360287471 \dots$$

To relate these logarithms to \log_2 we use some properties of logarithms:

1. $\log_b(b^y) = y$
2. $b^{\log_b y} = y$
3. $\log_b(xy) = \log_b x + \log_b y$
4. $\log_b x^y = y \log_b x$

We can apply this to the equation $2^{\frac{x}{1200}} = \frac{F_2}{F_1}$, to get:

$$\log_{10} \left(2^{\frac{x}{1200}} \right) = \log_{10} \left(\frac{F_2}{F_1} \right)$$

which simplifies using the last property above to

$$\frac{x}{1200} \log_{10} 2 = \log_{10} \left(\frac{F_2}{F_1} \right)$$

or simply

$$x = \frac{1200}{\log_{10} 2} \log_{10} \left(\frac{F_2}{F_1} \right)$$

Similarly, we perform the same sequence of steps using \ln instead of \log_{10} to obtain:

$$x = \frac{1200}{\ln 2} \ln \left(\frac{F_2}{F_1} \right)$$

For example, the cent measurement of the interval with frequency ratio $3/2$ is:

$$x = \frac{1200}{\ln 2} \ln \left(\frac{3}{2} \right) = 701.9550 \dots$$

and the cent measurement of the interval with frequency ratio $5/4$ is:

$$x = \frac{1200}{\ln 2} \ln \left(\frac{5}{4} \right) = 386.3137 \dots$$

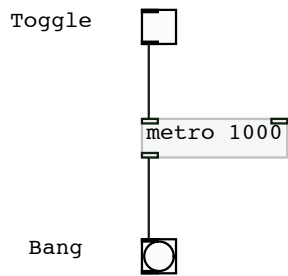
More on Frequency Ratios and Intervals

We discussed problem 1d from homework 2 on frequency ratios. Comments: If you don't know if $F_1 < F_2$ or $F_2 < F_1$, then the frequency ratio F_2/F_1 may come out with value larger than one or less than one. Similarly, the cent value will come out negative or positive respectively. In the problem, the two just intervals (a just fourth plus a just third) will come out slightly lower in frequency than the equal tempered intervals.

Pd Examples:

Example 1: metro with toggle and bang

The toggle box is like an ON/OFF switch. It is connected to the metro 1000 box on the left, or hot inlet. Note: It doesn't make a sound at this point, just outputs a bang on one second intervals.

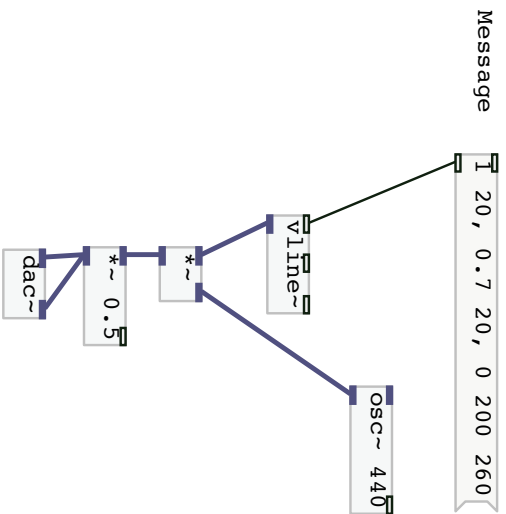


Use this patch to see how toggle can turn metro on and off, and how metro sends out a bang every 1000 milliseconds (ms). Edit 1000 to change the rate of the metronome.

Example 2: vline with osc, signal multiplication

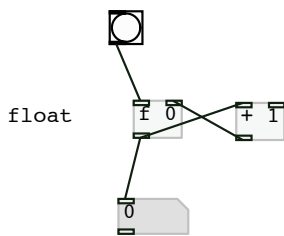
Vline is a signal output box, like osc. The output in either case is a function which you can think of as

a bunch of dots on a graph. The dots are coming out at the rate (called the sample rate) of 44100 per second. That's a lot of dots! These are all numbers which represent a graph such as an envelope function (vline) or a sine function (osc).



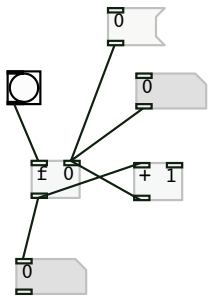
Use this one to play A440 with an envelope. The envelope ramps up to amplitude 1 in 20 ms, then down to 0.7 in another 20 ms, then sustains for 260 ms and then ramps down to 0 in 200 ms. So the total envelope time is 500 ms. Edit the times to change the envelope or edit the frequency 440 to change the pitch.

Example 3: float box



This patch uses a float box to store a variable number initialized to zero. The bang in the upper left or "hot" inlet pushes that number out through the lower outlet, where it is passed on to the number box and also to the plus one box's left or "hot" inlet.

The plus one box receives the number from the float box and adds one to it. Then it outputs the new value to the right inlet of the float box. The right inlet is called "cold" since it does not prompt an outlet. It simply stores that value as its current value without outputting it. When the next bang occurs, that value is pushed out.



In this patch we have added a message box with zero and a number box with zero to the cold inlet of the float. These can be used to reset the stored value of the float to zero by clicking on the message, or to any value by changing the number box value by clicking and holding then sliding up or down with the mouse. Note: the output in the lower number box can also be changed but this is overridden by any values coming from above.

Just Frequency Ratios and Scale Construction

We can place some of the frequency ratios that we have observed so far into a scale which takes on the

pattern of steps in a major scale:

We have introduced $3/2$ as the perfect fifth, $4/3$ as the perfect fourth, and $9/8$ as the ratio of the interval between the fourth and the fifth. We also call $9/8$ a just major whole tone. Next, we show how to construct the entire just diatonic scale starting with these frequency ratios.

Some of the other fractions with small numerator and denominator, that are between 1 and 2 are: $5/4$ and $5/3$. We call $5/4$ the just major third, and $5/3$ the just major sixth. In order of increasing size we can now place the following frequency ratios:

$$\frac{1}{1} \longrightarrow \frac{9}{8} \longrightarrow \frac{5}{4} \longrightarrow \frac{4}{3} \longrightarrow \frac{3}{2} \longrightarrow \frac{5}{3} \longrightarrow \frac{2}{1}$$

Next, we can also note the frequency ratios between each of the successive steps in this sequence. These are: $9/8$, $10/9$ (called a just minor whole tone), $16/15$, $9/8$, $10/9$, and $6/5$. (More on $6/5$ later - this will be the just minor third.) Since the last step of $6/5$ is larger than $9/8$, we could also break it up into smaller steps. For instance, a step of $9/8$ from the value $5/3$ gives us the new frequency ratio to insert: $15/8$. The step from $15/8$ to $2/1$ is then $16/15$, the just semitone. The sequence now looks like this, with the sequence of frequency ratios compared to the tonic ($1/1$) followed by the list of step sizes in parentheses:

The Just Diatonic Scale

$$\frac{1}{1} \longrightarrow \frac{9}{8} \longrightarrow \frac{5}{4} \longrightarrow \frac{4}{3} \longrightarrow \frac{3}{2} \longrightarrow \frac{5}{3} \longrightarrow \frac{15}{8} \longrightarrow \frac{2}{1}$$

$$\left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{16}{15}\right) \left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{9}{8}\right) \left(\frac{16}{15}\right).$$

We can also look at this from the viewpoint of cent measurement. The frequency ratios are computed based on the formulas:

$$2^{\frac{x}{1200}} = \frac{F_2}{F_1}$$

and

$$x = \frac{1200}{\ln 2} \ln \left(\frac{F_2}{F_1}\right).$$

The same scale then becomes:

The Just Diatonic Scale in Cents

$$0 \longrightarrow 203.91 \longrightarrow 386.31 \longrightarrow 498.04 \longrightarrow 701.96 \longrightarrow 884.36 \longrightarrow 1088.27 \longrightarrow 1200$$

$$(203.91) \quad (182.40) \quad (111.73) \quad (203.91) \quad (182.40) \quad (203.91) \quad (111.73)$$

For comparison, the equal tempered major (diatonic) scale with frequency ratios based on $2^{1/12}$ looks like:

The Equal-Tempered Diatonic Scale

$$\frac{1}{1} \longrightarrow 2^{2/12} \longrightarrow 2^{4/12} \longrightarrow 2^{5/12} \longrightarrow 2^{7/12} \longrightarrow 2^{9/12} \longrightarrow 2^{11/12} \longrightarrow 2^{12/12}$$

$$\left(2^{2/12}\right) \left(2^{2/12}\right) \left(2^{1/12}\right) \left(2^{2/12}\right) \left(2^{2/12}\right) \left(2^{2/12}\right) \left(2^{1/12}\right)$$

The same scale in cents looks like:

The Equal-Tempered Diatonic Scale in Cents

$$0 \longrightarrow 200 \longrightarrow 400 \longrightarrow 500 \longrightarrow 700 \longrightarrow 900 \longrightarrow 1100 \longrightarrow 1200$$

$$(200) (200) (100) (200) (200) (200) (100)$$

Introduction of Project on Instrument Sounds

An article on Mathematics of Musical Instruments was distributed. This article introduces some instruments of historical and acoustic interest, starting with the ancient Chinese flutes and the Scandinavian Willow flute, or overtone flute.

A wav file sample of Chinese flute sounds was displayed in Audacity. We also looked at the analysis-plot-spectrum function to find the fundamental frequency of some excerpts of the flute sound. Students will explore an instrument of their choice, find sound samples, import into Audacity, do spectrum analysis, and also simulate these frequencies in Pd. One requirement is that the sounds need to contain intervals which are outside the equal temperament model.

Next, we discussed the Just Diatonic and Pythagorean Diatonic Scales and compared these.

Note: In this scale there are two types of whole tones. We call them the *major* (9/8) and *minor* (10/9) whole tones.

The Just Diatonic Scale

$$\frac{1}{1} \longrightarrow \frac{9}{8} \longrightarrow \frac{5}{4} \longrightarrow \frac{4}{3} \longrightarrow \frac{3}{2} \longrightarrow \frac{5}{3} \longrightarrow \frac{15}{8} \longrightarrow \frac{2}{1}$$

$$\left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{16}{15}\right) \left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{9}{8}\right) \left(\frac{16}{15}\right).$$

In the Pythagorean Diatonic Scale, we eliminate the two different whole tones of the Just Scale, and use only $9/8$.

The Pythagorean Diatonic Scale

$$\frac{1}{1} \rightarrow \frac{9}{8} \rightarrow \frac{81}{64} \rightarrow \frac{4}{3} \rightarrow \frac{3}{2} \rightarrow \frac{27}{16} \rightarrow \frac{243}{128} \rightarrow \frac{2}{1}$$

$$\left(\frac{9}{8}\right) \left(\frac{9}{8}\right) \left(\frac{256}{243}\right) \left(\frac{9}{8}\right) \left(\frac{9}{8}\right) \left(\frac{9}{8}\right) \left(\frac{256}{243}\right).$$

Note: This scale agrees with the Just Diatonic Scale at not only the two ends $\frac{1}{1}$ and $\frac{2}{1}$ but also at $\frac{9}{8}$, $\frac{4}{3}$, and $\frac{3}{2}$. Next, we compute the difference between the two scales at the other pitches. For instance, we can ask how far apart are the frequencies $\frac{81}{64}$ and $\frac{5}{4}$? As a frequency ratio we find:

$$\frac{81/64}{5/4} = \frac{81}{64} \cdot \frac{4}{5} = \frac{81}{80}.$$

We can do the same at the other places where the scales disagree, and we find:

$$\frac{27/16}{5/3} = \frac{81}{80},$$

and

$$\frac{243/128}{16/15} = \frac{81}{80}.$$

In summary, the Pythagorean Diatonic Scale can be obtained from the Just Diatonic Scale by raising the third, sixth and seventh degrees of the scale by the frequency ratio: $\frac{81}{80}$. This small interval is called the *Syntonic Comma*. A comma is usually a very small interval which represents an adjustment or comparison between scale systems.

In cents the Syntonic Comma is about $1/5$ of a semitone:

$$\frac{1200}{\ln 2} \ln \left(\frac{81}{80}\right) = 21.5\dots$$

Decimal calculations for frequency ratios

Error propagation can be illustrated with the following examples:

If an approximation for $2^{1/12}$ is used, we can see how it affects the value of an interval on the piano. We can compare the values 1.059 and 1.059463. Let's compute the frequency ratio for one octave:

$$1.059^{12} = 1.989534, \quad 1.059463^{12} = 1.999997.$$

In cents we can measure these two frequency ratios:

$$\frac{1200}{\ln 2} \ln(1.989534) = 1190.92,$$

$$\frac{1200}{\ln 2} \ln(1.999997) = 1199.99.$$

So using the value 1.059 means we are already 10 cents off of the exact value for an octave, which should be 1200 cents. If we use the value 1.059463 for $2^{1/12}$, we are only off by 0.01 cents at the octave. These differences are significant for audio, since we can hear a difference of 10 cents, but certainly not 0.01 cents. (It is very difficult to hear a difference of 2 cents.)

Let's see how big the difference is if we go from the lowest to the highest notes on a piano:

$$1.059^{87} = 146.54, \quad 1.059463^{87} = 152.22.$$

In cents we can measure these two frequency ratios:

$$\frac{1200}{\ln 2} \ln(146.54) = 8634.18,$$

$$\frac{1200}{\ln 2} \ln(152.22) = 8700.02.$$

Note: the last value is almost exactly 8700 cents, which is 87 equal tempered semitones. The other approximation 8634.18 is almost 66 cents off of the true value. This is a very audible difference.

The Meantone Diatonic Scale

The first steps toward tempered scales, where irrational number approximations are used for the frequency ratios, are obtained by preserving the just major third, but making the whole step to be exactly half of this interval. This means we have two equal whole tone steps with ratio x :

$$\frac{1}{1} \xrightarrow{(x)} x \xrightarrow{(x)} \frac{5}{4}$$

which means that we need $x^2 = \frac{5}{4}$, which we can solve as:

$$x = \sqrt{\frac{5}{4}}.$$

This is the whole tone value for the mean-tone scale. The cent value is then:

$$\frac{1200}{\ln 2} \ln \left(\sqrt{\frac{5}{4}} \right) = 193.16$$

We can compare to our other whole tone cent values: Just major whole tone:

$$\frac{1200}{\ln 2} \ln \left(\frac{9}{8} \right) = 203.91,$$

just minor whole tone:

$$\frac{1200}{\ln 2} \ln \left(\frac{10}{9} \right) = 182.40,$$

equal tempered whole tone:

$$200.$$

The Meantone Diatonic Scale

The first steps toward tempered scales, where irrational number approximations are used for the frequency ratios, are obtained by preserving the just major third, but making the whole step to be exactly half of this interval. This means we have two equal whole tone steps with ratio r :

$$\frac{1}{1} \xrightarrow{(r)} r \xrightarrow{(r)} \frac{5}{4}$$

which means that we need $r^2 = \frac{5}{4}$, which we can solve as:

$$r = \sqrt{\frac{5}{4}}.$$

This is the whole tone value for the mean-tone scale. The cent value is then:

$$\frac{1200}{\ln 2} \ln \left(\sqrt{\frac{5}{4}} \right) = 193.16$$

We can compare to our other whole tone cent values: Just major whole tone:

$$\frac{1200}{\ln 2} \ln \left(\frac{9}{8} \right) = 203.91,$$

just minor whole tone:

$$\frac{1200}{\ln 2} \ln \left(\frac{10}{9} \right) = 182.40,$$

equal tempered whole tone:

$$200.$$

In the meantone scale we try to preserve the major thirds and create an equal whole step based on half of the major third. This means that our whole step will need to have frequency ratio $\sqrt{\frac{5}{4}}$, since

$$\sqrt{\frac{5}{4}} \sqrt{\frac{5}{4}} = \frac{5}{4}.$$

Next, we observe that there are three places in the just diatonic scale where the $5/4$ major third occurs.

(The Just Diatonic Scale)

$$\frac{1}{1} \rightarrow \frac{9}{8} \rightarrow \frac{5}{4} \rightarrow \frac{4}{3} \rightarrow \frac{3}{2} \rightarrow \frac{5}{3} \rightarrow \frac{15}{8} \rightarrow \frac{2}{1}$$

$$\left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{16}{15}\right) \left(\frac{9}{8}\right) \left(\frac{10}{9}\right) \left(\frac{9}{8}\right) \left(\frac{16}{15}\right).$$

In this scale, the major third is identified as the combination of the major and minor whole tones

$$\frac{9}{8} \frac{10}{9} = \frac{5}{4}.$$

This occurs from $\frac{4}{3}$ to $\frac{5}{3}$ and also from $\frac{3}{2}$ to $\frac{15}{8}$. So we start by inserting the whole steps as follows, with the half-step or semitone, indicated with an s :

$$\frac{1}{1} \rightarrow \sqrt{\frac{5}{4}} \rightarrow \frac{5}{4} \rightarrow ? \rightarrow ? \rightarrow ? \rightarrow ? \rightarrow \frac{2}{1}$$

$$\left(\sqrt{\frac{5}{4}}\right) \left(\sqrt{\frac{5}{4}}\right) (s) \left(\sqrt{\frac{5}{4}}\right) \left(\sqrt{\frac{5}{4}}\right) \left(\sqrt{\frac{5}{4}}\right) (s)$$

Next, we can solve for the semitone s by requiring that all of these consecutive steps make up one complete octave:

$$\sqrt{\frac{5}{4}} \cdot \sqrt{\frac{5}{4}} \cdot s \cdot \sqrt{\frac{5}{4}} \cdot \sqrt{\frac{5}{4}} \cdot \sqrt{\frac{5}{4}} \cdot s = 2$$

or equivalently:

$$\sqrt{\frac{5}{4}}^5 \cdot s^2 = 2$$

which yields:

$$s^2 = \frac{2}{\sqrt{\frac{5}{4}}^5} \quad \text{and} \quad s = \sqrt{\frac{2}{\sqrt{\frac{5}{4}}^5}} = \frac{\sqrt{2}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}.$$

Finally, we can state the entire scale as:

The Meantone Major Scale

$$\frac{1}{1} \rightarrow \sqrt{\frac{5}{4}} \rightarrow \frac{5}{4} \rightarrow \frac{\sqrt{2}}{\left(\frac{5}{4}\right)^{\frac{1}{4}}} \rightarrow \sqrt{2} \left(\frac{5}{4}\right)^{\frac{1}{4}} \rightarrow \sqrt{2} \left(\frac{5}{4}\right)^{\frac{3}{4}} \rightarrow \sqrt{2} \left(\frac{5}{4}\right)^{\frac{5}{4}} \rightarrow \frac{2}{1}$$

$$\left(\sqrt{\frac{5}{4}}\right) \quad \left(\sqrt{\frac{5}{4}}\right) \quad \left(\frac{\sqrt{2}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right) \quad \left(\sqrt{\frac{5}{4}}\right) \quad \left(\sqrt{\frac{5}{4}}\right) \quad \left(\sqrt{\frac{5}{4}}\right) \quad \left(\frac{\sqrt{2}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right)$$

Sinusoids and phase shift

A general sine function can be expressed as

$$y = A \sin(Bt + C)$$

where $|A|$ is the amplitude, and the period is given by $2\pi/B$. If the time t is measured in seconds, then the frequency is given by $B/2\pi$ cycles per second, or Hz (Herz).

Next we look at the phase shift. First, we can rewrite the equation as

$$y = A \sin(Bt + C) = A \sin(B(t + C/B)),$$

which is a shift of the function

$$y = A \sin(Bt)$$

by the amount $|C/B|$ to the left or right. In particular, the direction depends on the sign of C/B : left if $C/B > 0$ and right if $C/B < 0$.

Explanations for Homework exercises

In question 1 you are to construct a scale, or geometric sequence, which is based on a certain interval or frequency ratio and a starting frequency. The frequency ratio is obtained by dividing the interval up into equal pieces. The basic example for this process is the equal tempered chromatic scale, which divides the octave into 12 equal sized pieces called semitones. This is obtained by observing that the frequency ratio r must then have the property that

$$r^{12} = 2$$

which gives $r = 2^{1/12}$.

We can mimic this process for any interval and any number of steps. For example, we could ask for 13 equal divisions of the octave and obtain the step size $r = 2^{1/13}$.

Or we could ask for 10 equal divisions of the just perfect fifth, which would give the step size r which satisfies $r^{10} = \frac{3}{2}$, or

$$r = \frac{3^{1/10}}{2}.$$

This value of r can also be measured in cents by:

$$x = \frac{1200}{\ln 2} \ln \left(\frac{3^{1/10}}{2} \right) = 70.1955 \dots$$

In question 2 make sure that you are comparing the frequency ratios that give the degree of the scale, not the step up from the previous degree. So, for instance the third degree of the scale for both the just and mean-tone scales is given by the frequency ratio $5/4$, which is the multiple of the first degree of the scale. So if the scale starts at 440, then the third degree has frequency 550. The cent value is then also the same in both scales at the third degree. You need to compare the cent values at the other degrees to find which are the closest in value.

In questions 3 the scale is constructed so that there are 5 equal divisions of the just perfect fourth $4/3$ frequency ratio, which occurs from the first to the fourth degrees, and also from the fifth to the eighth degrees (since the fifth $3/2$ up to the octave 2 is obtained with the multiplier $4/3$.) This gives one type of semitone. The other type is obtained by making two equal divisions of the whole step between the just fourth and fifth degrees of the scale.

In question 4 solve for each of A , B , and C separately, using the appropriate piece of information. For instance, the frequency or the period will allow you to solve for B . The amplitude allows you to solve for A . The other piece of information will allow you to solve for C .

For 5 and 6 note that a , d , s and r are the time intervals for the Attack, Decay, Sustain and Release portions of the envelope. When you write it as a Pd message, make sure that you use the correct syntax with commas and spaces.

Prime limit for scales

We can factor the numerator and denominator of any fraction in prime powers. If the fraction is reduced, then there are no factors common to both the numerator and denominator.

A set of fractions has a “prime limit”, which simply means that all primes that occur in the numerator or denominator of the reduced forms of these fractions are less than or equal to the prime limit.

For example, the set of fractions $\{\frac{3}{2}, \frac{5}{3}, 2\}$ has prime limit 5 since the only primes occurring are 2, 3, and 5. We also say that this is a “5-limit” set of fractions.

If the frequency ratios present in a scale are all fractions, then we can also define the prime limit of such a scale.

For example, the Just Diatonic Scale has prime limit 5. The Just Chromatic Scale also has prime limit 5, as we can see by simply factoring the numerators and denominators of the fractions:

The Just (Natural) Chromatic Scale

$$\frac{1}{1} \longrightarrow \frac{16}{15} \longrightarrow \frac{9}{8} \longrightarrow \frac{6}{5} \longrightarrow \frac{5}{4} \longrightarrow \frac{4}{3} \longrightarrow \frac{64}{45} \longrightarrow \frac{3}{2} \longrightarrow \frac{8}{5} \longrightarrow \frac{5}{3} \longrightarrow \frac{16}{9} \longrightarrow \frac{15}{8} \longrightarrow \frac{2}{1}$$

To construct scales with higher prime limits, one usually also chooses some limit on the size of the numerator and denominator. For example, in the Just Diatonic Scale, the largest numerator or denominator is 16. In the Just Chromatic Scale, the largest is 64. We call the maximum size of the numerator or denominator of a fraction the *height* of the fraction.

For example, $\frac{16}{15}$ has height 16, whereas $\frac{64}{45}$ has height 64.

The Just Diatonic Scale is a height 16 5-limit scale. The Just Chromatic Scale is a height 64 5-limit scale.

In the homework you are asked to construct a height 16 7-limit scale.

Comments on the distribution of prime numbers:

There is no formula which gives exactly the list of prime numbers less than some real number x . However, it was proved in 1896 that the number of such prime numbers decreases with x and follows the approximate graph of what is called the “Prime Number Function”:

$$\pi(x) \equiv \frac{x}{\log x}.$$

Addition of functions

Two functions can be added together by drawing the two graphs on the same coordinate system, then adding the y -values of points together. It helps to do this first for some special values, for instance where one of the graphs is zero or reaches a maximum.

SciLab examples

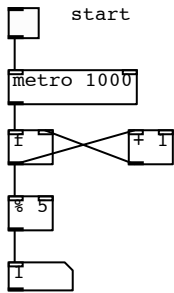
SciLab is installed on the Von Neumann macs. Find it with the puffin icon in the applications folder. From within SciLab you can execute a file with commands in it, or you can type the commands directly into the SciLab window. For now, it is easier to execute a given file with correct syntax already entered. Some sample files are on the web site.

To execute a file, such as the graph of a sine function, select file-execute and then choose the file from your computer. The file should have the .sci extension.

More Pd examples

Example 4: mod function: %

Modular arithmetic uses remainders.

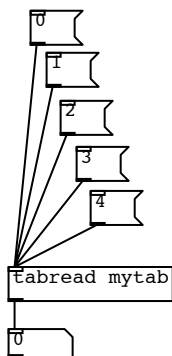


The box `% 5` takes input from the `f` box and gives the remainder after division by 5. This means that the number box at the bottom will read 0 after it reaches 4, and then it will continue to cycle through 0 to 4.

Example 5: indexed tables with Pd

`table mytab 5` creates a table called `mytab` that can store 5 things

`mytab 0 7 -2 3 8 13` when this message is clicked it puts the five numbers 7 -2 3 8 13 into `mytab`, with indices:
0 1 2 3 4

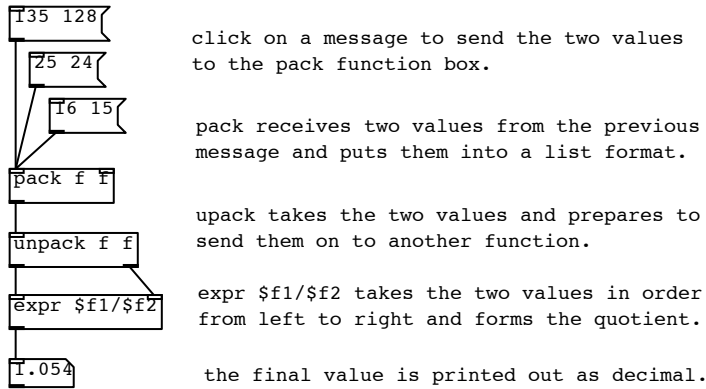


click on a message to send an index value to `tabread` and then watch the value with that index in `mytab` come out the bottom.

Example 6: pack and unpack

This example is used to convert from the fractional form of a frequency ratio to the decimal approximation form. Note: the number box does not display all of the digits that are actually saved in this number.

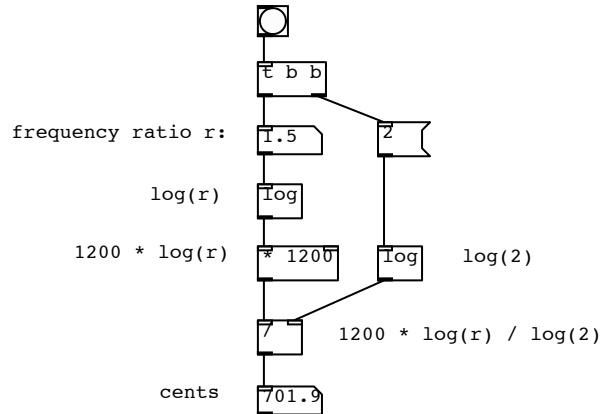
There are about 10 digits of accuracy in addition to the 4 that are displayed.



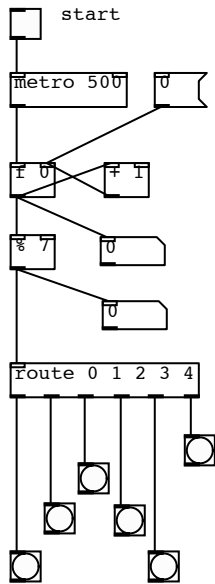
Example 7: trigger box, and frequency ratio to cent conversion

this computes the cent value of a frequency ratio

`l b b` sends out 2 bangs in order from right to left



Example 8: route function



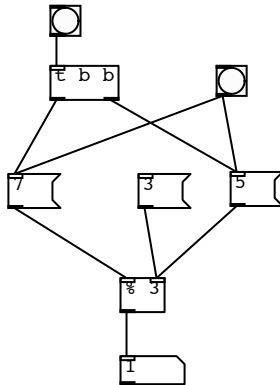
This patch illustrates the route function. Route is a like a branching function which takes the inlet and sends it through one of several different outlets according to what that inlet is.

In this example, the possible inlet values that are recognized by route are 0,1,2,3,4. Notice that this is 5 values, however there are in fact 6 outlets. The sixth outlet is used if the inlet does not match anything from the list 0,1,2,3,4.

If you push the start button you will see the bang boxes below light up on the half second. The first 5 will respond to the numbers 0,1,2,3,4 and then the sixth one will respond to the other numbers 5 and 6. Since we are using % 7, the values will cycle through 0 to 6.

More Pd examples

Example 8: route function



This patch illustrates the order of bangs sent out. The two bangs on the top can be used to send out two more bangs to the messages with numbers 7 and 5. However, they have a different effect on the output at the bottom because they have different orders.

In particular: by clicking the number message 3, we are putting the 3 into the cold inlet of the % box. This means that the number 3 is the modulus, or the number which we divide by in order to find the remainder. The remainder after division by 3 is what comes out at the bottom. This is why we see 1 at the bottom.

If we push the 5 message, the modulus is changed to 5 and so now when we push 7 the output is 2.

On the other hand, we can use the upper right bang to hit both the 5 and the 7. The desired effect would be to get 2 out the bottom again. To test this, push 3 again and then push the upper right bang. The output is 1. Why?

The output is 1 because the double bang is being sent out with the order left then right. This way the left bang first hits the 7 which goes all the way through the hot inlet of % and pushes out the remainder 1 since the modulus is still 3. The other bang is too late with the 5, so it stores the modulus as 5 only after the 7 has gone through. At this point we can again push the 7 and see that we get a remainder of 2 since the modulus is now 5.

Finally, we can run the same test on the trigger box, which should send out the two bangs in the order from right to left. So, first push 3, then test 7 to get 1. Now push the top bang above trigger and get 2. This time the 5 gets loaded first, before the 7 goes through and so the remainder is indeed 2.

Tones as Fourier series or sums of sine functions

Each musical note can be modeled mathematically as a sum of sine functions. A basic example is that of a plucked string where the basic or fundamental frequency has overtones which are integer multiples of the fundamental frequency. For example, we might have a fundamental frequency of 220 Hz and a sum of sine functions such as:

$$f(t) = A_0 \sin(220 \cdot 2\pi t) + A_1 \sin(440 \cdot 2\pi t) + A_2 \sin(660 \cdot 2\pi t) + \dots$$

where the amplitudes are typically decreasing:

$$A_0 > A_1 > A_2 > \dots$$

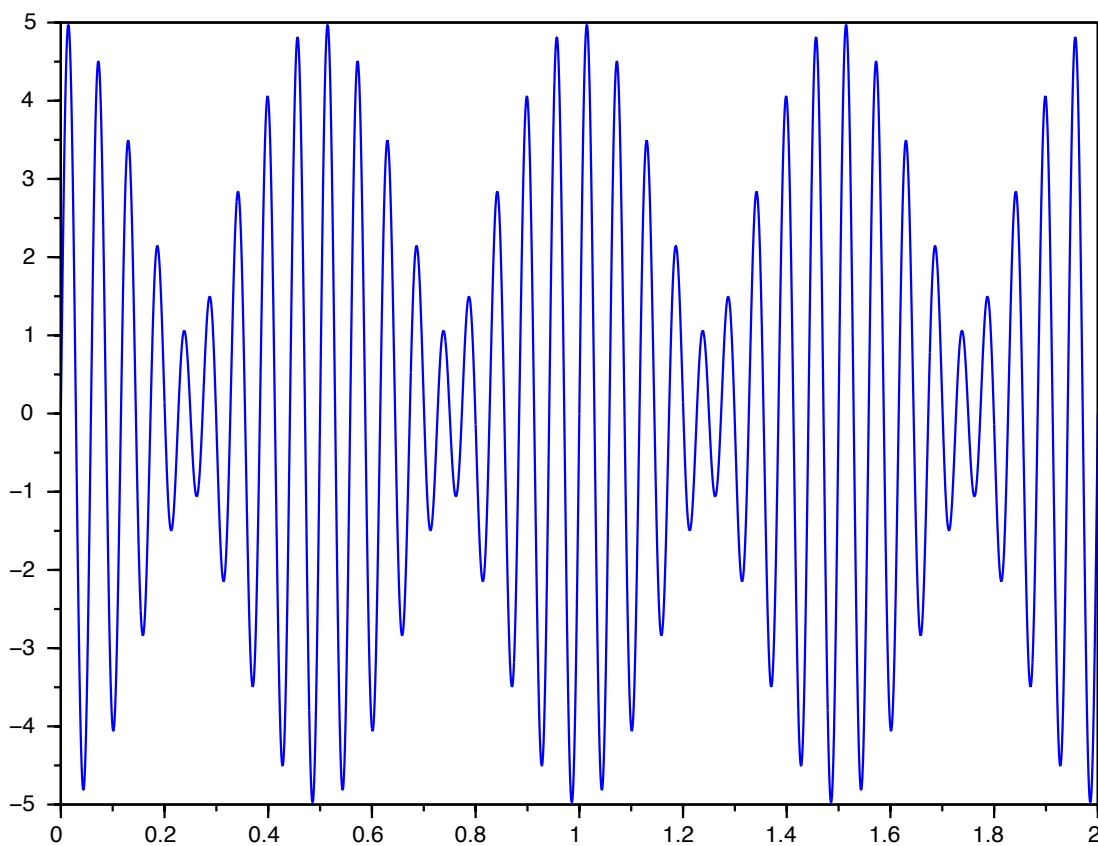
and the sum can be taken as finite when the volume drops below our hearing range.

In reality the situation is more complex, in particular:

- sounds may have *inharmonic* overtones which are not integer multiples of the fundamental
- each overtone frequency may have its own envelope

Beats

Graph of sine sum with beats:



$$y = 3 \sin(36\pi t) + 2 \sin(32\pi t)$$

Note: the first term has amplitude 3 and frequency 18 and the second term has amplitude 2 and frequency 16. Based on the difference of the frequencies, we expect to see 2 beats per second, which is verified in the graph.

Questions on Pd project 2

How do we integrate the tabread and route functions?

The route function receives values, which have some specific meaning to your patch. For instance, these could be the values that you store in your table, which are used to represent different types of semitone or whole tone. You might call those 1, 2 and 3 for example. According to which value goes into route, if it is recognized by route then it goes out a specific outlet. If it is not recognized by route then it will be sent out the last outlet.

Indices are read in by the tabread function, which are like markers for where the values are in your table. If the 0-spot in your table has the value 7 in there, then if you send in 0 to tabread it will push a 7 out its outlet.

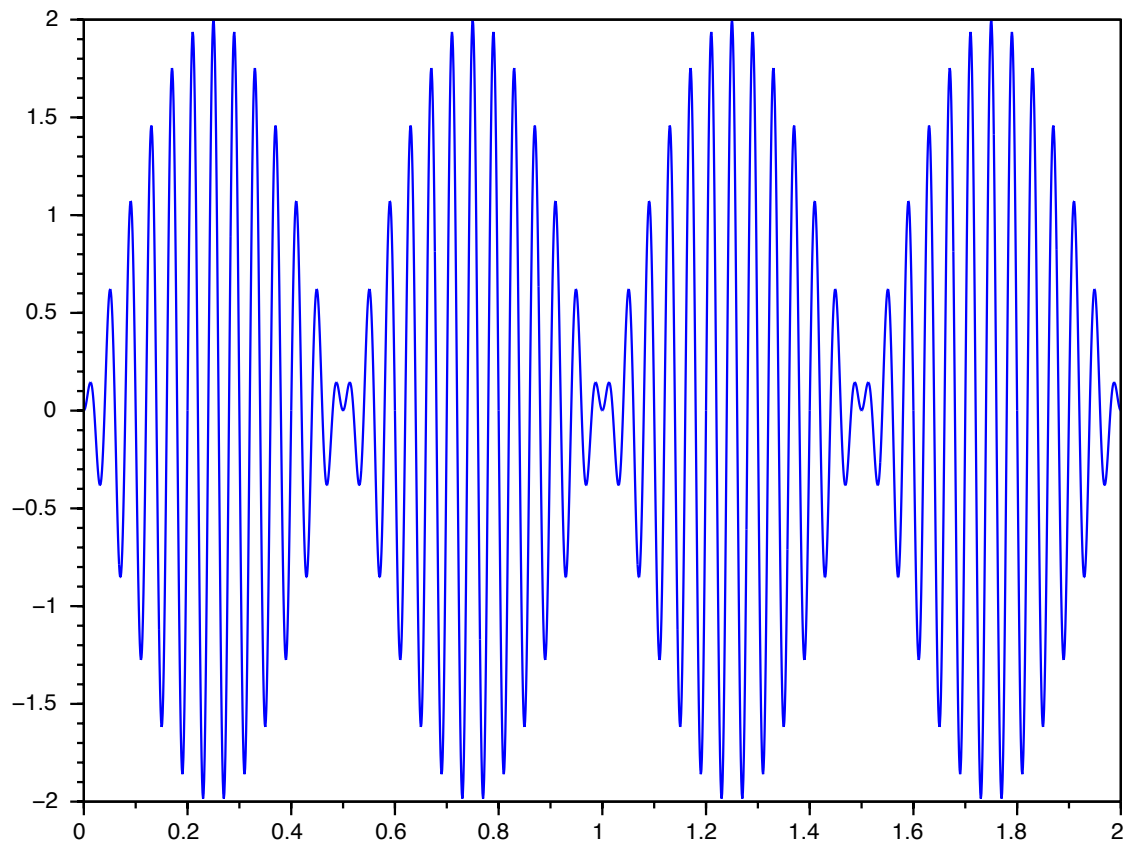
The values that come out can then be used to bang a message which contains the fraction, numerator and denominator, which represents a frequency ratio. Those message can then be converted into decimal form in order to multiply a frequency and get a new one which is higher by that particular ratio, or interval.

Remember that a number can serve as a value and also as a bang.

More about Beats

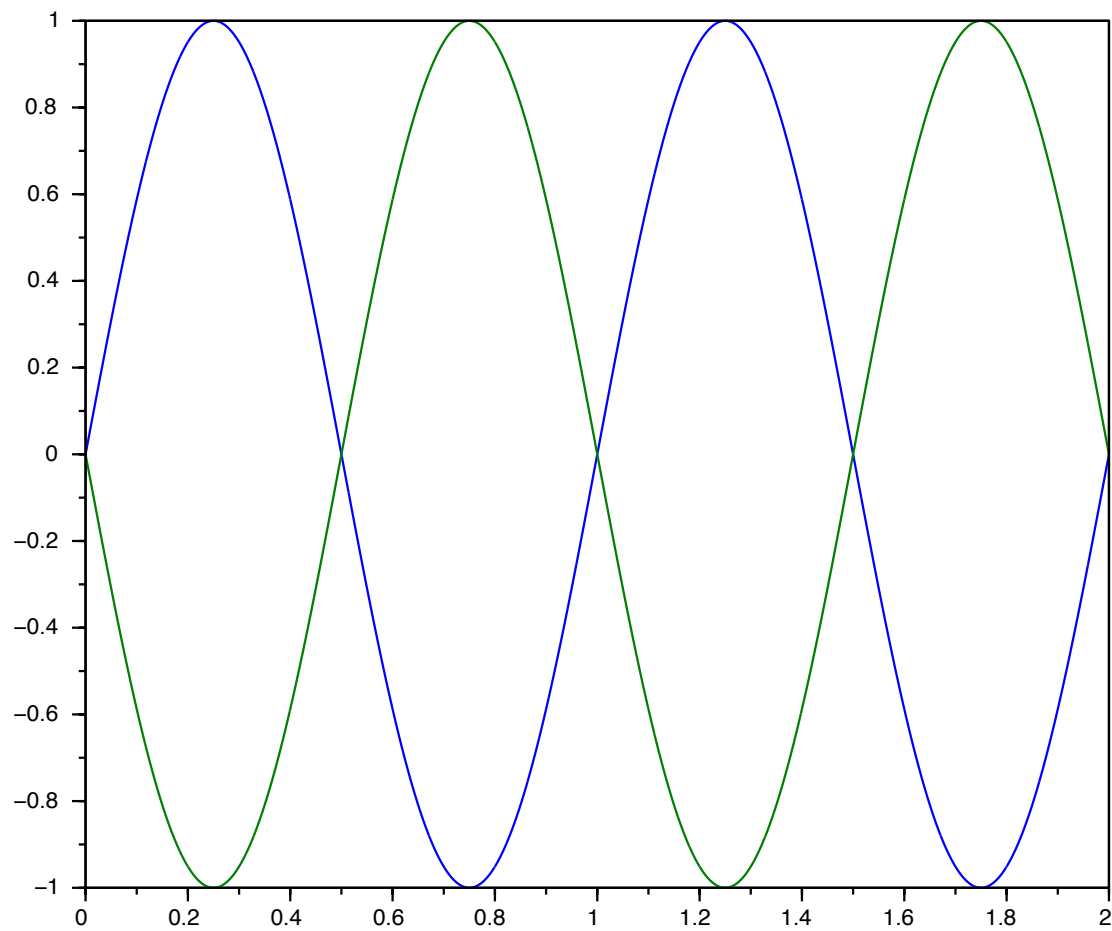
In the previous lecture we saw that a sum of two sine function with very similar frequencies results in a graph which has a pattern of amplitude oscillations, which we call beats. We also heard the beats in Pd and noticed that we could predict the number of beats as the difference in the frequencies. We will now take another approach to beats which first presents the same type of phenomenon in the product of two sine curves. But in this case one of the sine functions has very small frequency and the other is much larger.

Graph of sine product with beats:



$$y = \sin(2\pi t) \cdot 2 \sin(50\pi t)$$

Note: the first factor has amplitude 1 and frequency 1 and the second factor has amplitude 2 and frequency 25. The first one behaves like an envelope function for the second which has higher frequency. An envelope function is typically positive, but the graphs of the negative and positive versions are drawn in order to account for oscillations of the sine or cosine graph. If we graph both of these for the function $y = \sin(2\pi t)$ we get: ($y = \pm \sin(2\pi t)$)



$$y = \pm \sin(2\pi t)$$

Next time, we will pursue the connection between the two versions of beats, which will make use of a trigonometric identity called appropriately the product to sum identity.

Basic facts about sine and cosine functions

Before we can understand the product to sum identity, we need to work a bit more precisely with the definitions and properties of the sine and cosine functions.

These two functions are derived from the unit circle in the x,y plane, given by the equation

$$x^2 + y^2 = 1.$$

Now, in order to define $\sin t$ and $\cos t$ for a specific real number input t , we first treat the number t as an arc length. We measure all arc lengths on the unit circle starting at the point $(1, 0)$. For positive t we measure in the counterclockwise direction, and for negative t we measure clockwise. The end point of the arc length t we can label as (x_t, y_t) . Finally, we define the functions:

$$\sin t = y_t \quad \text{and} \quad \cos t = x_t.$$

From this definition, and the knowledge of the circumference 2π of the unit circle, we can immediately get some values. If $t = 0$ then there is no arc length so the starting and ending point are the same point $(1, 0)$, and we have:

$$\sin 0 = 0 \quad \text{and} \quad \cos 0 = 1.$$

Similarly, if the arc length is $t = 2\pi$ then the starting and ending point are the same point $(1, 0)$, and we have:

$$\sin 2\pi = 0 \quad \text{and} \quad \cos 2\pi = 1.$$

Also, if the arc length is $t = \pi$ then the end point is $(-1, 0)$ and we have:

$$\sin \pi = 0 \quad \text{and} \quad \cos \pi = -1.$$

If the arc length is $t = \pi/2$ then the end point is $(0, 1)$ and we have:

$$\sin \frac{\pi}{2} = 1 \quad \text{and} \quad \cos \frac{\pi}{2} = 0.$$

Similarly, if the arc length is $t = 3\pi/2$ then the end point is $(0, -1)$ and we have:

$$\sin \frac{3\pi}{2} = -1 \quad \text{and} \quad \cos \frac{3\pi}{2} = 0.$$

Next, to compute values which do not have end point on the axes, we need to use some reference triangles. A reference triangle is one which fits into the unit circle, so its hypotenuse must have length one. Further, one of its sides must be on the x -axis, and one must make a right angle with the x -axis. The basic reference triangles are the 30 – 60 and the 45 triangles.

We can find the lengths of the sides of these two triangles easily. The 45 triangle has two angles of 45 degrees, and one 90 degree or right angle. If we label the two equal sides with length a , then we have from Pythagoras' theorem:

$$a^2 + a^2 = 1^2$$

which means that $2a^2 = 1$, or $a^2 = 1/2$, and so we have

$$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

For the 30 – 60 triangle, we can first observe that two of these side by side make up a triangle which has three angles of 60 degrees, and is hence an equilateral triangle. So all the side have the same length, which we can call c . We also see that one of the other sides has length $\frac{1}{2}c$. Then by Pythagoras' Theorem again, the remaining side length b must satisfy:

$$\left(\frac{1}{2}c\right)^2 + b^2 = c^2$$

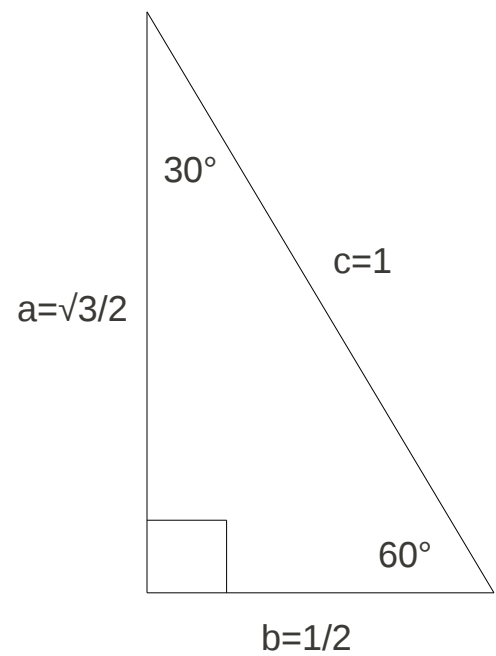
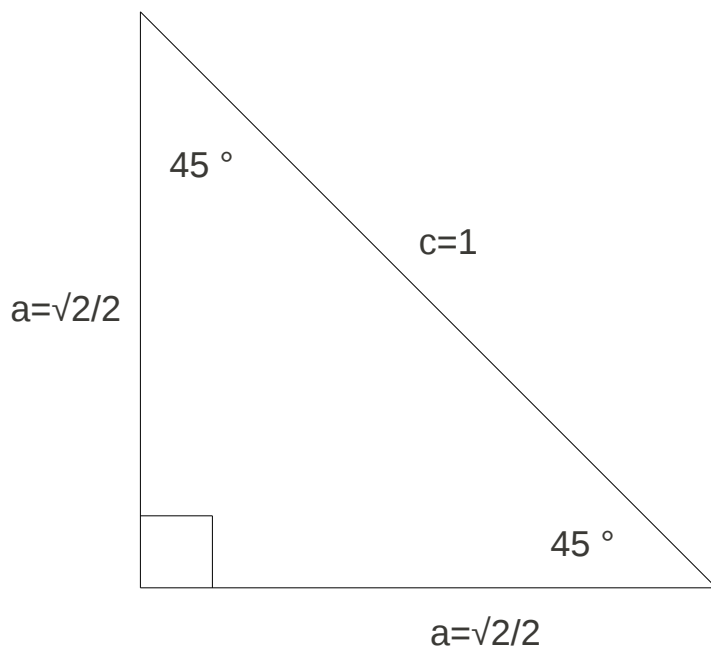
which means

$$b^2 = c^2 - \left(\frac{1}{2}c\right)^2 = c^2 - \frac{1}{4}c^2 = \frac{3}{4}c^2$$

and we have

$$b = \frac{\sqrt{3}}{2}c.$$

Now if we set $c = 1$, so that this triangle fits into the unit circle, we have sides of length $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$.



Trig Identities and Rotation matrices

The most basic trig identity comes from the definitions that we saw last time using the unit circle. Since a point (x, y) on the unit circle can be labelled with coordinates

$$(x, y) = (\cos t, \sin t)$$

where t is an arc length measured from the starting point $(1, 0)$ counterclockwise for positive t , clockwise for negative t , and since all points on the unit circle satisfy the equation

$$x^2 + y^2 = 1$$

it must be the case that

$$(\cos t)^2 + (\sin t)^2 = 1.$$

This can also be written:

$$\sin^2 t + \cos^2 t = 1.$$

The next important identities are the sum and difference identities. These can be derived in various ways. One of the most interesting is to use rotation matrices, although the full derivation in this context would require more linear algebra. But we at least show the connection since it can also be a nice way to remember the identities.

The following array of numbers based on the *sine* and *cosine* of a number θ is called the rotation matrix:

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

For two such matrices we can also define a product. The important point here (which is the part that follows from more linear algebra) is that the product actually represents two successive rotations, so it is natural to expect that the result is a rotation first by the angle a and then by the angle b .

$$R_{a+b} = R_a R_b$$

So this means that:

$$\begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

The last step is to use the general 2×2 matrix multiplication:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Combining these we have:

$$\begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{pmatrix} = \begin{pmatrix} \cos a \cos b - \sin a \sin b & -\sin a \cos b - \cos a \sin b \\ \sin a \cos b + \cos a \sin b & \cos a \cos b - \sin a \sin b \end{pmatrix}$$

Then, by equating the corresponding matrix entries, we get:

Sum Identity for cosine:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

Sum Identity for sine:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

Next, we use the fact that

$$\cos(-x) = \cos x \quad \text{and} \quad \sin(-x) = -\sin x$$

to arrive at:

Difference Identity for sine:

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

Difference Identity for cosine:

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Product to Sum Identity for cosines:

$$\cos a \cos b = \frac{1}{2} [\cos(a + b) + \cos(a - b)]$$

Product to Sum Identity for sines:

$$\sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

Mixed Product to Sum Identity for sines and cosines:

$$\sin a \cos b = \frac{1}{2} [\sin(a + b) + \sin(a - b)]$$

$$\cos a \sin b = \frac{1}{2} [\sin(a + b) - \sin(a - b)]$$

Sum to Product Identity for sines:

$$\sin A + \sin B = 2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)$$

Sum to Product Identity for cosines:

$$\cos A + \cos B = 2 \cos \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)$$

Change of variables in Sum to Product Identities:

$$A = a + b, \quad B = a - b, \quad a = \frac{1}{2}(A + B), \quad b = \frac{1}{2}(A - B)$$

In matrix form:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$