47. Consider the system

$$\begin{cases} x_1 y_2^2 - 2x_2 y_3 = 1 \\ x_1 y_1^5 + x_2 y_2 - 4y_2 y_3 = -9 . \\ x_2 y_1 + 3x_1 y_3^2 = 12 \end{cases}$$

(a) Show that, near the point $(x_1, x_2, y_1, y_2, y_3) = (1, 0, -1, 1, 2)$, it is possible to solve for y_1, y_2, y_3 in terms of x_1, x_2.

(b) From the result of part (a), we may consider y_1, y_2, y_3 to be functions of x_1 and x_2. Use implicit differentiation and the chain rule to evaluate $\frac{\partial y_1}{\partial x_1}(1, 0)$, $\frac{\partial y_2}{\partial x_1}(1, 0)$, and $\frac{\partial y_3}{\partial x_1}(1, 0)$.

48. Consider the equations that relate cylindrical and Cartesian coordinates in $\mathbf{R}^3$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta . \\ z = z \end{cases}$$

(a) Near which points of $\mathbf{R}^3$ can we solve for r, θ, and z in terms of the Cartesian coordinates?

(b) Explain the geometry behind your answer in part (a).

49. Recall that the equations relating spherical and Cartesian coordinates in $\mathbf{R}^3$ are

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta . \\ z = \rho\cos\varphi \end{cases}$$

(a) Near which points of $\mathbf{R}^3$ can we solve for ρ, φ, and θ in terms of x, y, and z?

(b) Describe the geometry behind your answer in part (a).

2.7 Newton's Method (optional)

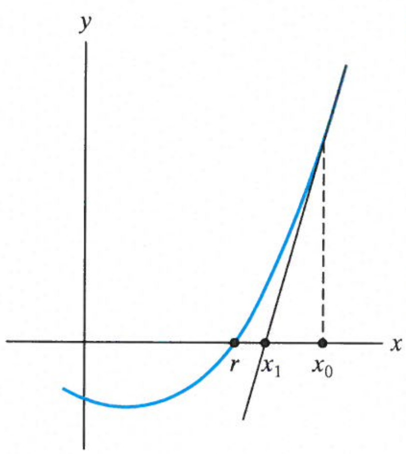

Figure 2.75 The tangent line to $y = f(x)$ at $(x_0, f(x_0))$ crosses the x-axis at $x = x_1$.

When you studied single-variable calculus, you may have learned a method, known as **Newton's method** (or the **Newton–Raphson method**), for approximating the solution to an equation of the form $f(x) = 0$, where $f\colon X \subseteq \mathbf{R} \to \mathbf{R}$ is a differentiable function. Here's a reminder of how the method works.

We wish to find a number r such that $f(r) = 0$. To approximate r, we make an initial guess x_0 for r and, in general, we expect to find that $f(x_0) \neq 0$. So next we look at the tangent line to the graph of f at $(x_0, f(x_0))$. (See Figure 2.75.) Since the tangent line approximates the graph of f near $(x_0, f(x_0))$, we can find where the tangent line crosses the x-axis. The crossing point $(x_1, 0)$ will generally be closer to $(r, 0)$ than $(x_0, 0)$ is, so we take x_1 as a revised and improved approximation to the root r of $f(x) = 0$.

To find x_1, we begin with the equation of the tangent line

$$y = f(x_0) + f'(x_0)(x - x_0),$$

then set $y = 0$ to find where this line crosses the x-axis. Thus, we solve the equation

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$

for x_1 to find that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Once we have x_1, we can start the process again using x_1 in place of x_0 and produce what we hope will be an even better approximation x_2 via the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Indeed, we may iterate this process and define x_k recursively by

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} \qquad k = 1, 2, \ldots \tag{1}$$

and thereby produce a sequence of numbers $x_0, x_1, \ldots, x_k, \ldots$.

It is not always the case that the sequence $\{x_k\}$ converges. However, when it does, it must converge to a root of the equation $f(x) = 0$. To see this, let $L = \lim_{k\to\infty} x_k$. Then we also have $\lim_{k\to\infty} x_{k-1} = L$. Taking limits in formula (1), we find

$$L = L - \frac{f(L)}{f'(L)},$$

which immediately implies that $f(L) = 0$. Hence, L is a root of the equation.

Now that we have some understanding of derivatives in the multivariable case, we turn to the generalization of Newton's method for solving systems of n equations in n unknowns. We may write such a system as

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ f_2(x_1, \ldots, x_n) = 0 \\ \quad\vdots \\ f_n(x_1, \ldots, x_n) = 0 \end{cases}. \tag{2}$$

We consider the map $\mathbf{f}\colon X \subseteq \mathbf{R}^n \to \mathbf{R}^n$ defined as $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ (i.e., $\mathbf{f}$ is the map whose component functions come from the equations in (2). The domain X of $\mathbf{f}$ may be taken to be the set where all the component functions are defined.) Then to solve system (2) means to find a vector $\mathbf{r} = (r_1, \ldots, r_n)$ such that $\mathbf{f}(\mathbf{r}) = \mathbf{0}$. To approximate such a vector $\mathbf{r}$, we may, as in the single-variable case, make an initial guess $\mathbf{x}_0$ for what $\mathbf{r}$ might be. If $\mathbf{f}$ is differentiable, then we know that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is approximated by the equation

$$\mathbf{y} = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

(Here we think of $\mathbf{f}(\mathbf{x}_0)$ and the vectors $\mathbf{x}$ and $\mathbf{x}_0$ as $n \times 1$ matrices.) Then we set $\mathbf{y}$ equal to $\mathbf{0}$ to find where this approximating function is zero. Thus, we solve the matrix equation

$$\mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}_1 - \mathbf{x}_0) = \mathbf{0} \tag{3}$$

for $\mathbf{x}_1$ to give a revised approximation to the root $\mathbf{r}$. Evidently (3) is equivalent to

$$D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}_1 - \mathbf{x}_0) = -\mathbf{f}(\mathbf{x}_0). \tag{4}$$

To continue our argument, suppose that $D\mathbf{f}(\mathbf{x}_0)$ is an invertible $n \times n$ matrix, meaning that there is a second $n \times n$ matrix $[D\mathbf{f}(\mathbf{x}_0)]^{-1}$ with the property that $[D\mathbf{f}(\mathbf{x}_0)]^{-1}D\mathbf{f}(\mathbf{x}_0) = D\mathbf{f}(\mathbf{x}_0)[D\mathbf{f}(\mathbf{x}_0)]^{-1} = I_n$, the $n \times n$ identity matrix. (See Exercises 20 and 30–38 in §1.6.) Then we may multiply equation (4) on the left by $[D\mathbf{f}(\mathbf{x}_0)]^{-1}$ to obtain

$$I_n(\mathbf{x}_1 - \mathbf{x}_0) = -[D\mathbf{f}(\mathbf{x}_0)]^{-1}\mathbf{f}(\mathbf{x}_0).$$

Since $I_n A = A$ for any $n \times k$ matrix A, this last equation implies that

$$\mathbf{x}_1 = \mathbf{x}_0 - [D\mathbf{f}(\mathbf{x}_0)]^{-1}\mathbf{f}(\mathbf{x}_0). \tag{5}$$

As we did in the one-variable case of Newton's method, we may iterate formula (5) to define recursively a sequence $\{\mathbf{x}_k\}$ of *vectors* by

$$\boxed{\mathbf{x}_k = \mathbf{x}_{k-1} - [D\mathbf{f}(\mathbf{x}_{k-1})]^{-1}\mathbf{f}(\mathbf{x}_{k-1})} \tag{6}$$

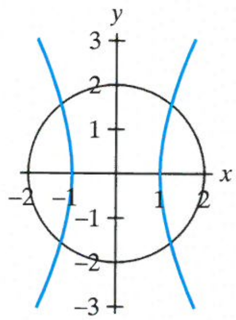

Figure 2.76 Finding the intersection points of the circle $x^2 + y^2 = 4$ and the hyperbola $4x^2 - y^2 = 4$ in Example 1.

Note the similarity between formulas (1) and (6). Moreover, just as in the case of formula (1), although the sequence $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots\}$ may not converge, if it does, it must converge to a root of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. (See Exercise 4.)

EXAMPLE 1 Consider the problem of finding the intersection points of the circle $x^2 + y^2 = 4$ and the hyperbola $4x^2 - y^2 = 4$. (See Figure 2.76.) Analytically, we seek simultaneous solutions to the two equations

$$x^2 + y^2 = 4 \quad \text{and} \quad 4x^2 - y^2 = 4,$$

or, equivalently, solutions to the system

$$\begin{cases} x^2 + y^2 - 4 = 0 \\ 4x^2 - y^2 - 4 = 0 \end{cases}. \tag{7}$$

To use Newton's method, we define a function $\mathbf{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $\mathbf{f}(x, y) = (x^2 + y^2 - 4, 4x^2 - y^2 - 4)$ and try to approximate solutions to the vector equation $\mathbf{f}(x, y) = (0, 0)$. We may begin with any initial guess, say,

$$\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and then produce successive approximations $\mathbf{x}_1, \mathbf{x}_2, \ldots$ to a solution using formula (6). In particular, we have

$$D\mathbf{f}(x, y) = \begin{bmatrix} 2x & 2y \\ 8x & -2y \end{bmatrix}.$$

Note that $\det D\mathbf{f}(x, y) = -20xy$. You may verify (see Exercise 36 in §1.6) that

$$[D\mathbf{f}(x, y)]^{-1} = \frac{1}{-20xy}\begin{bmatrix} -2y & -2y \\ -8x & 2x \end{bmatrix} = \begin{bmatrix} \dfrac{1}{10x} & \dfrac{1}{10x} \\ \dfrac{2}{5y} & -\dfrac{1}{10y} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - [D\mathbf{f}(x_{k-1}, y_{k-1})]^{-1}\mathbf{f}(x_{k-1}, y_{k-1})$$

$$= \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \begin{bmatrix} \dfrac{1}{10x_{k-1}} & \dfrac{1}{10x_{k-1}} \\ \dfrac{2}{5y_{k-1}} & -\dfrac{1}{10y_{k-1}} \end{bmatrix} \begin{bmatrix} x_{k-1}^2 + y_{k-1}^2 - 4 \\ 4x_{k-1}^2 - y_{k-1}^2 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \begin{bmatrix} \dfrac{5x_{k-1}^2 - 8}{10x_{k-1}} \\ \dfrac{5y_{k-1}^2 - 12}{10y_{k-1}} \end{bmatrix} = \begin{bmatrix} x_{k-1} - \dfrac{5x_{k-1}^2 - 8}{10x_{k-1}} \\ y_{k-1} - \dfrac{5y_{k-1}^2 - 12}{10y_{k-1}} \end{bmatrix}.$$

Beginning with $x_0 = y_0 = 1$, we have

$$x_1 = 1 - \frac{5 \cdot 1^2 - 8}{10 \cdot 1} = 1.3 \qquad y_1 = 1 - \frac{5 \cdot 1^2 - 12}{10 \cdot 1} = 1.7$$

$$x_2 = 1.3 - \frac{5(1.3)^2 - 8}{10(1.3)} = 1.265385 \quad y_2 = 1.7 - \frac{5(1.7)^2 - 12}{10(1.7)} = 1.555882, \quad \text{etc.}$$

It is also easy to hand off the details of the computation to a calculator or a computer. One finds the following results:

k	x_k	y_k
0	1	1
1	1.3	1.7
2	1.26538462	1.55588235
3	1.26491115	1.54920772
4	1.26491106	1.54919334
5	1.26491106	1.54919334

Thus, it appears that, to eight decimal places, an intersection point of the curves is (1.26491106, 1.54919334).

In this particular example, it is not difficult to find the solutions to (7) exactly. We add the two equations in (7) to obtain

$$5x^2 - 8 = 0 \qquad \Longleftrightarrow \qquad x^2 = \tfrac{8}{5}.$$

Thus, $x = \pm\sqrt{8/5}$. If we substitute these values for x into the first equation of (7), we obtain

$$\tfrac{8}{5} + y^2 - 4 = 0 \qquad \Longleftrightarrow \qquad y^2 = \tfrac{12}{5}.$$

Hence, $y = \pm\sqrt{12/5}$. Therefore, the four intersection points are

$$\left(\sqrt{\frac{8}{5}}, \sqrt{\frac{12}{5}}\right), \quad \left(-\sqrt{\frac{8}{5}}, \sqrt{\frac{12}{5}}\right), \quad \left(-\sqrt{\frac{8}{5}}, -\sqrt{\frac{12}{5}}\right), \quad \left(\sqrt{\frac{8}{5}}, -\sqrt{\frac{12}{5}}\right).$$

Since $\sqrt{8/5} \approx 1.264911064$ and $\sqrt{12/5} \approx 1.54919334$, we see that Newton's method provided us with an accurate approximate solution very quickly. ◆

EXAMPLE 2 We use Newton's method to find solutions to the system

$$\begin{cases} x^3 - 5x^2 + 2x - y + 13 = 0 \\ x^3 + x^2 - 14x - y - 19 = 0 \end{cases}. \tag{8}$$

As in the previous example, we define $\mathbf{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $\mathbf{f}(x, y) = (x^3 - 5x^2 + 2x - y + 13, x^3 + x^2 - 14x - y - 19)$. Then

$$D\mathbf{f}(x, y) = \begin{bmatrix} 3x^2 - 10x + 2 & -1 \\ 3x^2 + 2x - 14 & -1 \end{bmatrix},$$

so that $\det D\mathbf{f}(x, y) = 12x - 16$ and

$$[D\mathbf{f}(x, y)]^{-1} = \begin{bmatrix} -\dfrac{1}{12x - 16} & \dfrac{1}{12x - 16} \\ \dfrac{-3x^2 - 2x + 14}{12x - 16} & \dfrac{-3x^2 - 10x + 2}{12x - 16} \end{bmatrix}.$$

Thus, formula (6) becomes

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \begin{bmatrix} -\dfrac{1}{12x_{k-1} - 16} & \dfrac{1}{12x_{k-1} - 16} \\ \dfrac{-3x_{k-1}^2 - 2x_{k-1} + 14}{12x_{k-1} - 16} & \dfrac{-3x_{k-1}^2 - 10x_{k-1} + 2}{12x_{k-1} - 16} \end{bmatrix}$$

$$\times \begin{bmatrix} x_{k-1}^3 - 5x_{k-1}^2 + 2x_{k-1} - y_{k-1} + 13 \\ x_{k-1}^3 + x_{k-1}^2 - 14x_{k-1} - y_{k-1} - 19 \end{bmatrix}$$

$$= \begin{bmatrix} x_{k-1} - \dfrac{6x_{k-1}^2 - 16x_{k-1} - 32}{12x_{k-1} - 16} \\ y_{k-1} - \dfrac{3x_{k-1}^4 - 16x_{k-1}^3 - 14x_{k-1}^2 + 82x_{k-1} - 8y_{k-1} + 6x_{k-1}y_{k-1} + 72}{6x_{k-1} - 8} \end{bmatrix}.$$

This is the formula we iterate to obtain approximate solutions to (8).

If we begin with $\mathbf{x}_0 = (x_0, y_0) = (8, 10)$, then the successive approximations $\mathbf{x}_k$ quickly converge to $(4, 5)$, as demonstrated in the table below.

k	x_k	y_k
0	8	10
1	5.2	−98.2
2	4.1862069	−2.7412414
3	4.00607686	4.82161865
4	4.00000691	4.99981073
5	4.00000000	5.00000000
6	4.00000000	5.00000000

If we begin instead with $\mathbf{x}_0 = (50, 60)$, then convergence is, as you might predict, somewhat slower (although still quite rapid):

k	x_k	y_k
0	50	60
1	25.739726	−57257.438
2	13.682211	−7080.8238
3	7.79569757	−846.58548
4	5.11470969	−86.660453
5	4.1643023	−1.6486813
6	4.00476785	4.86119425
7	4.00000425	4.99988349
8	4.00000000	5.00000000
9	4.00000000	5.00000000

On the other hand, if we begin with $\mathbf{x}_0 = (-2, 12)$, then the sequence of points generated converges to a different solution, namely, $(-4/3, -25/27)$:

k	x_k	y_k
0	-2	12
1	-1.4	1.4
2	-1.3341463	-0.903122
3	-1.3333335	-0.9259225
4	-1.3333333	-0.9259259
5	-1.3333333	-0.9259259

In fact, when a system of equations has multiple solutions, it is not always easy to predict to which solution a given starting vector $\mathbf{x}_0$ will converge under Newton's method (if, indeed, there is convergence at all). ◆

Finally, we make two remarks. First, if at any stage of the iteration process the matrix $D\mathbf{f}(\mathbf{x}_k)$ fails to be invertible (i.e., $[D\mathbf{f}(\mathbf{x}_k)]^{-1}$ does not exist), then formula (6) cannot be used. One way to salvage the situation is to make a different choice of initial vector $\mathbf{x}_0$ in the hope that the sequence $\{\mathbf{x}_k\}$ that it generates will not involve any noninvertible matrices. Second, we note that if, at any stage, $\mathbf{x}_k$ is *exactly* a root of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, then formula (6) will not change it. (See Exercise 7).

2.7 Exercises

1. Use Newton's method with initial vector $\mathbf{x}_0 = (1, -1)$ to approximate the real solution to the system

$$\begin{cases} y^2e^x = 3 \\ 2ye^x + 10y^4 = 0 \end{cases}.$$

2. In this problem, you will use Newton's method to estimate the locations of the points of intersection of the ellipses having equations $3x^2 + y^2 = 7$ and $x^2 + 4y^2 = 8$.

(a) Graph the ellipses and use your graph to give a very rough estimate (x_0, y_0) of the point of intersection that lies in the first quadrant.

(b) Denote the *exact* point of intersection in the first quadrant by (X, Y). Without solving, argue that the other points of intersection must be $(-X, Y)$, $(X, -Y)$, and $(-X, -Y)$.

(c) Now use Newton's method with your estimate (x_0, y_0) in part (a) to approximate the first quadrant intersection point (X, Y).

(d) Solve for the intersection points exactly, and compare your answer with your approximations.

3. This problem concerns the determination of the points of intersection of the two curves with equations $x^3 - 4y^3 = 1$ and $x^2 + 4y^2 = 2$.

(a) Graph the curves and use your graph to give rough estimates for the points of intersection.

(b) Now use Newton's method with different initial estimates to approximate the intersection points.

4. Consider the sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots,$ where, for $k \geq 1$, the vector $\mathbf{x}_k$ is defined by the Newton's method recursion formula (6) given an initial "guess" $\mathbf{x}_0$ at a root of the equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. (Here we assume that $\mathbf{f}\colon X \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is a differentiable function.) By imitating the argument in the single-variable case, show that if the sequence $\{\mathbf{x}_k\}$ converges to a vector $\mathbf{L}$ and $D\mathbf{f}(\mathbf{L})$ is an invertible matrix, then $\mathbf{L}$ must satisfy $\mathbf{f}(\mathbf{L}) = \mathbf{0}$.

5. This problem concerns the Newton's method iteration in Example 1.

(a) Use initial vector $\mathbf{x}_0 = (-1, 1)$ and calculate the successive approximations $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, etc. To what solution of the system of equations (7) do the approximations converge?

(b) Repeat part (a) with $\mathbf{x}_0 = (1, -1)$. Repeat again with $\mathbf{x}_0 = (-1, -1)$.

(c) Comment on the results of parts (a) and (b) and whether you might have predicted them. Describe the results in terms of Figure 2.76.

6. Consider the Newton's method iteration in Example 2.
   (a) Use initial vector $\mathbf{x}_0 = (1.4, 10)$ and calculate the successive approximations $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, etc. To what solution of the system of equations (8) do the approximations converge?
   (b) Repeat part (a) with $\mathbf{x}_0 = (1.3, 10)$.
   (c) In Example 2 we saw that $(4, 5)$ was a solution of the given system of equations. Is $(1.3, 10)$ closer to $(4, 5)$ or to the limiting point of the sequence you calculated in part (b)?
   (d) Comment on your observations in part (c). What do these observations suggest about how easily you can use the initial vector $\mathbf{x}_0$ to predict the value of $\lim_{k\to\infty} \mathbf{x}_k$ (assuming that the limit exists)?

7. Suppose that at some stage in the Newton's method iteration using formula (6), we obtain a vector $\mathbf{x}_k$ that is an exact solution to the system of equations (2). Show that all the subsequent vectors $\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \ldots$ are equal to $\mathbf{x}_k$. Hence, if we happen to obtain an exact root via Newton's method, we will retain it.

8. Suppose that $\mathbf{f}: X \subseteq \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable and that we write $\mathbf{f}(x, y) = (f(x, y), g(x, y))$. Show that formula (6) implies that, for $k \geq 1$,

$$x_k = x_{k-1} - \frac{f(x_{k-1}, y_{k-1})g_y(x_{k-1}, y_{k-1}) - g(x_{k-1}, y_{k-1})f_y(x_{k-1}, y_{k-1})}{f_x(x_{k-1}, y_{k-1})g_y(x_{k-1}, y_{k-1}) - f_y(x_{k-1}, y_{k-1})g_x(x_{k-1}, y_{k-1})}$$

$$y_k = y_{k-1} - \frac{g(x_{k-1}, y_{k-1})f_x(x_{k-1}, y_{k-1}) - f(x_{k-1}, y_{k-1})g_x(x_{k-1}, y_{k-1})}{f_x(x_{k-1}, y_{k-1})g_y(x_{k-1}, y_{k-1}) - f_y(x_{k-1}, y_{k-1})g_x(x_{k-1}, y_{k-1})}.$$

9. As we will see in Chapter 4, when looking for maxima and minima of a differentiable function $F: X \subseteq \mathbf{R}^n \to \mathbf{R}$, we need to find the points where $DF(x_1, \ldots, x_n) = [0 \cdots 0]$, called **critical points** of F. Let $F(x, y) = 4\sin(xy) + x^3 + y^3$. Use Newton's method to approximate the critical point that lies near $(x, y) = (-1, -1)$.

10. Consider the problem of finding the intersection points of the sphere $x^2 + y^2 + z^2 = 4$, the circular cylinder $x^2 + y^2 = 1$, and the elliptical cylinder $4y^2 + z^2 = 4$.
    (a) Use Newton's method to find one of the intersection points. By choosing a different initial vector $\mathbf{x}_0 = (x_0, y_0, z_0)$, approximate a second intersection point. (Note: You may wish to use a computer algebra system to determine appropriate inverse matrices.)
    (b) Find all the intersection points exactly by means of algebra and compare with your results in part (a).

True/False Exercises for Chapter 2

1. The component functions of a vector-valued function are vectors.

2. The domain of $\mathbf{f}(x, y) = \left(x^2 + y^2 + 1, \dfrac{3}{x+y}, \dfrac{x}{y}\right)$ is $\{(x, y) \in \mathbf{R}^2 \mid y \neq 0, x \neq y\}$.

3. The range of $\mathbf{f}(x, y) = \left(x^2 + y^2 + 1, \dfrac{3}{x+y}, \dfrac{x}{y}\right)$ is $\{(u, v, w) \in \mathbf{R}^3 \mid u \geq 1\}$.

4. The function $\mathbf{f}: \mathbf{R}^3 - \{(0, 0, 0)\} \to \mathbf{R}^3$, $\mathbf{f}(\mathbf{x}) = 2\mathbf{x}/\|\mathbf{x}\|$ is one-one.

5. The graph of $x = 9y^2 + z^2/4$ is a paraboloid.

6. The graph of $z + x^2 = y^2$ is a hyperboloid.

7. The level set of a function $f(x, y, z)$ is either empty or a surface.

8. The graph of any function of two variables is a level set of a function of three variables.

9. The level set of any function of three variables is the graph of a function of two variables.

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2} = 1.$

11. If $f(x, y) = \begin{cases} \dfrac{y^4 - x^4}{x^2 + y^2} & \text{when } (x, y) \neq (0, 0) \\ 2 & \text{when } (x, y) = (0, 0) \end{cases}$, then f is continuous.

12. If $f(x, y)$ approaches a number L as $(x, y) \to (a, b)$ along all lines through (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

13. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ exists and is finite, then $\mathbf{f}$ is continuous at $\mathbf{a}$.

14. $f_x(a, b) = \lim_{x \to a} \dfrac{f(x, b) - f(a, b)}{x - a}$.

15. If $f(x, y, z) = \sin y$, then $\nabla f(x, y, z) = \cos y$.

16. If $\mathbf{f}\colon \mathbf{R}^3 \to \mathbf{R}^4$ is differentiable, then $D\mathbf{f}(\mathbf{x})$ is a 3×4 matrix.

17. If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$.

18. If $\mathbf{f}$ is continuous at $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$.

19. If all partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ of a function $f(x_1, \ldots, x_n)$ exist at $\mathbf{a} = (a_1, \ldots, a_n)$, then f is differentiable at $\mathbf{a}$.

20. If $\mathbf{f}\colon \mathbf{R}^4 \to \mathbf{R}^5$ and $\mathbf{g}\colon \mathbf{R}^4 \to \mathbf{R}^5$ are both differentiable at $\mathbf{a} \in \mathbf{R}^4$, then $D(\mathbf{f} - \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) - D\mathbf{g}(\mathbf{a})$.

21. There's a function f of class C^2 such that $\dfrac{\partial f}{\partial x} = y^3 - 2x$ and $\dfrac{\partial f}{\partial y} = y - 3xy^2$.

22. If the second-order partial derivatives of f exist at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

23. If $w = F(x, y, z)$ and $z = g(x, y)$ where F and g are differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial g}{\partial x}.$$

24. The tangent plane to $z = x^3/(y + 1)$ at the point $(-2, 0, -8)$ has equation $z = 12x + 8y + 16$.

25. The plane tangent to $xy/z^2 = 1$ at $(2, 8, -4)$ has equation $4x + y + 2z = 8$.

26. The plane tangent to the surface $x^2 + xye^z + y^3 = 1$ at the point $(2, -1, 0)$ is parallel to the vector $3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

27. $D_{\mathbf{j}} f(x, y, z) = \dfrac{\partial f}{\partial y}$.

28. $D_{-\mathbf{k}} f(x, y, z) = \dfrac{\partial f}{\partial z}$.

29. If $f(x, y) = \sin x \cos y$ and $\mathbf{v}$ is a unit vector in $\mathbf{R}^2$, then $0 \le D_{\mathbf{v}} f\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right) \le \dfrac{\sqrt{2}}{2}$.

30. If $\mathbf{v}$ is a unit vector in $\mathbf{R}^3$ and $f(x, y, z) = \sin x - \cos y + \sin z$, then

$$-\sqrt{3} \le D_{\mathbf{v}} f(x, y, z) \le \sqrt{3}.$$

Miscellaneous Exercises for Chapter 2

1. Let $\mathbf{f}(\mathbf{x}) = (\mathbf{i} + \mathbf{k}) \times \mathbf{x}$.

(a) Write the component functions of $\mathbf{f}$.

(b) Describe the domain and range of $\mathbf{f}$.

2. Let $\mathbf{f}(\mathbf{x}) = \mathrm{proj}_{3\mathbf{i}-2\mathbf{j}+\mathbf{k}}\mathbf{x}$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Describe the domain and range of $\mathbf{f}$.

(b) Write the component functions of $\mathbf{f}$.

3. Let $f(x, y) = \sqrt{xy}$.

(a) Find the domain and range of f.

(b) Is the domain of f open or closed? Why?

4. Let $g(x, y) = \sqrt{\dfrac{x}{y}}$.

(a) Determine the domain and range of g.

(b) Is the domain of g open or closed? Why?

5. Figure 2.77 shows the graphs of six functions $f(x, y)$ and plots of the collections of their level curves in some order. Complete the following table by matching each function in the table with its graph and plot of its level curves.

Function $f(x, y)$	Graph (uppercase letter)	Level curves (lowercase letter)
$f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$		
$f(x, y) = \sin\sqrt{x^2 + y^2}$		
$f(x, y) = (3y^2 - 2x^2)e^{-x^2-2y^2}$		
$f(x, y) = y^3 - 3x^2y$		
$f(x, y) = x^2y^2e^{-x^2-2y^2}$		
$f(x, y) = ye^{-x^2-y^2}$		

6. Consider the function $f(x, y) = 2 + \ln(x^2 + y^2)$.

(a) Sketch some level curves of f. Give at least those at heights, 0, 1, and 2. (It will probably help if you give a few more.)

(b) Using part (a) or otherwise, give a rough sketch of the graph of $z = f(x, y)$.

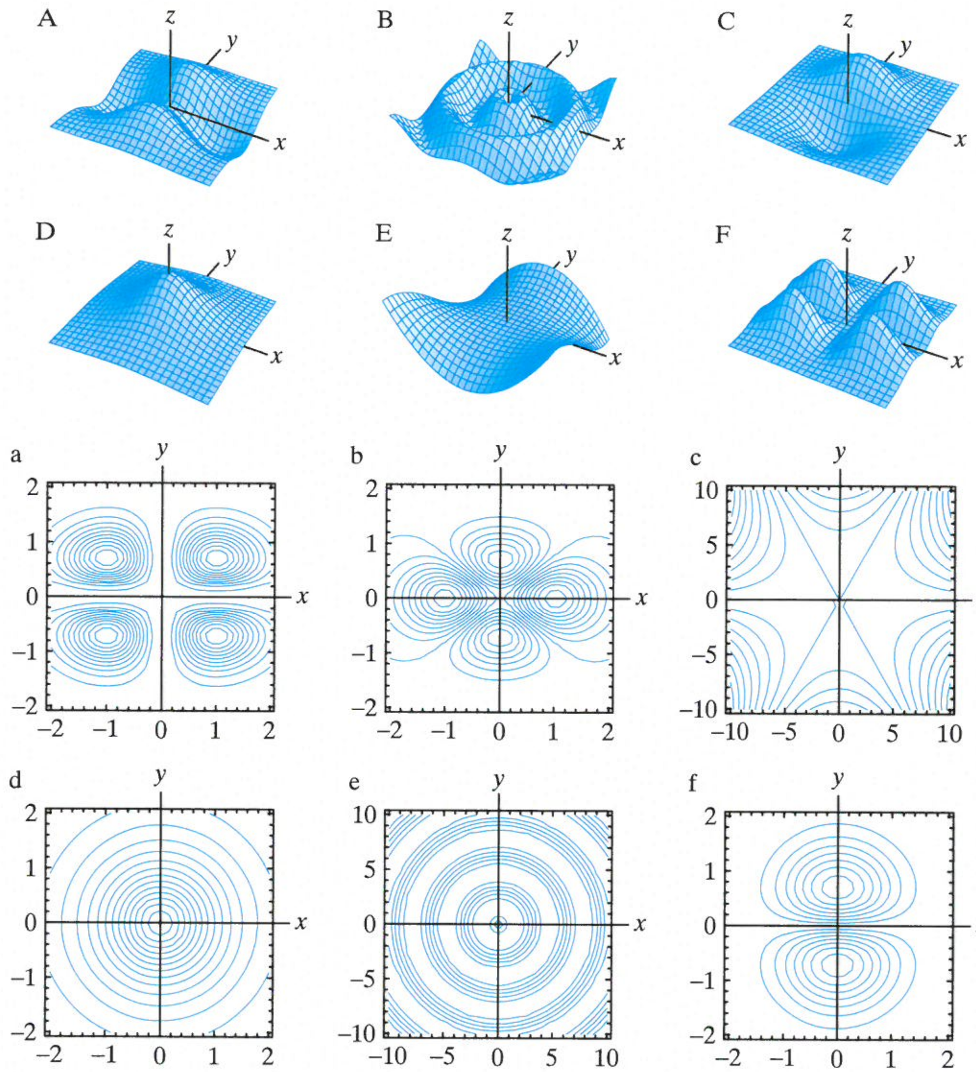

Figure 2.77 Figures for Exercise 5.

7. Use polar coordinates to evaluate

$$\lim_{(x,y)\to(0,0)} \frac{yx^2 - y^3}{x^2 + y^2}.$$

8. This problem concerns the function

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

(a) Use polar coordinates to describe this function.

(b) Using the polar coordinate description obtained in part (a), give some level curves for this function.

(c) Prepare a rough sketch of the graph of f.

(d) Determine $\lim_{(x,y)\to(0,0)} f(x, y)$, if it exists.

(e) Is f continuous? Why or why not?

9. Let

$$F(x, y) = \begin{cases} \dfrac{xy(xy + x^2)}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Show that the function $g(x) = F(x, 0)$ is continuous at $x = 0$. Show that the function $h(y) = F(0, y)$ is continuous at $y = 0$. However, show that F fails to be continuous at $(0, 0)$. (Thus, continuity in each variable separately does not necessarily imply continuity of the function.)

10. Suppose $f: U \subseteq \mathbf{R}^n \to \mathbf{R}$ is not defined at a point $\mathbf{a} \in \mathbf{R}^n$ but is defined for all $\mathbf{x}$ near $\mathbf{a}$. In other words, the domain U of f includes, for some $r > 0$, the set $B_r = \{\mathbf{x} \in \mathbf{R}^n \mid 0 < \|\mathbf{x} - \mathbf{a}\| < r\}$. (The set B_r is just an open ball of radius r centered at $\mathbf{a}$ with the point

a deleted.) Then we say $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = +\infty$ if $f(\mathbf{x})$ grows without bound as $\mathbf{x} \to \mathbf{a}$. More precisely, this means that given any $N > 0$ (no matter how large), there is some $\delta > 0$ such that if $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ (i.e., if $\mathbf{x} \in B_r$), then $f(\mathbf{a}) > N$.

(a) Using intuitive arguments or the preceding technical definition, explain why $\lim_{x\to 0} 1/x^2 = \infty$.

(b) Explain why

$$\lim_{(x,y)\to(1,3)} \frac{2}{(x-1)^2 + (y-3)^2} = \infty.$$

(c) Formulate a definition of what it means to say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = -\infty.$$

(d) Explain why

$$\lim_{(x,y)\to(0,0)} \frac{1-x}{xy^4 - y^4 + x^3 - x^2} = -\infty.$$

Exercises 11–17 involve the notion of windchill temperature—see Example 7 in §2.1, and refer to the table of windchill values on page 85.

11. (a) Find the windchill temperature when the air temperature is 25 °F and the windspeed is 10 mph.

(b) If the windspeed is 20 mph, what air temperature causes a windchill temperature of −15 °F?

12. (a) If the air temperature is 10 °F, estimate (to the nearest unit) what windspeed would give a windchill temperature of −5 °F.

(b) Do you think your estimate in part (a) is high or low? Why?

13. At a windspeed of 30 mph and air temperature of 35 °F, estimate the rate of change of the windchill temperature with respect to air temperature if the windspeed is held constant.

14. At a windspeed of 15 mph and air temperature of 25 °F, estimate the rate of change of the windchill temperature with respect to windspeed.

15. Windchill tables are constructed from empirically derived formulas for heat loss from an exposed surface. Early experimental work of P. A. Siple and C. F. Passel,[4] resulted in the following formula:

$$W = 91.4 + (t - 91.4)(0.474 + 0.304\sqrt{s} - 0.0203s).$$

Here W denotes windchill temperature (in degrees Fahrenheit), t the air temperature (for $t < 91.4$ °F), and s the windspeed in miles per hour (for $s \geq 4$ mph).[5]

(a) Compare your answers in Exercises 11 and 12 with those computed directly from the Siple formula just mentioned.

(b) Discuss any differences you observe between your answers to Exercises 11 and 12 and your answers to part (a).

(c) Why is it necessary to take $t < 91.4$ °F and $s \geq 4$ mph in the Siple formula? (Don't look for a purely mathematical reason; think about the model.)

16. Recent research led the United States National Weather Service to employ a new formula for calculating windchill values beginning November 1, 2001. In particular, the table on page 85 was constructed from the formula

$$W = 35.74 + 0.621t - 35.75s^{0.16} + 0.4275ts^{0.16}.$$

Here, as in the Siple formula of Exercise 15, W denotes windchill temperature (in degrees Fahrenheit), t the air temperature (for $t \leq 50$ °F), and s the windspeed in miles per hour (for $s \geq 3$ mph).[6] Compare your answers in Exercises 13 and 14 with those computed directly from the National Weather Service formula above.

17. In this problem you will compare graphically the two windchill formulas given in Exercises 15 and 16.

(a) If $W_1(s, t)$ denotes the windchill function given by the Siple formula in Exercise 15 and $W_2(s, t)$ the windchill function given by the National Weather Service formula in Exercise 16, graph the curves $y = W_1(s, 40)$ and $y = W_2(s, 40)$ on the same set of axes. (Let s vary between 3 and 120 mph.) In addition, graph other pairs of curves $y = W_1(s, t_0)$, $y = W_2(s, t_0)$ for other values of t_0. Discuss what your results tell you about the two windchill formulas.

(b) Now graph pairs of curves $y = W_1(s_0, t)$, $y = W_2(s_0, t)$ for various constant values s_0 for windspeed. Discuss your results.

(c) Finally, graph the surfaces $z = W_1(s, t)$ and $z = W_2(s, t)$ and comment.

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[4] "Measurements of dry atmospheric cooling in subfreezing temperatures," *Proc. Amer. Phil. Soc.*, **89** (1945), 177–199.

[5] From Bob Rilling, Atmospheric Technology Division, National Center for Atmospheric Research (NCAR), "Calculating Windchill Values," February 12, 1996. Found online at http://www.atd.ucar.edu/homes/rilling/wc_formula.html (July 31, 2010).

[6] NOAA, National Weather Service, Office of Climate, Water, and Weather Services, "NWS Wind Chill Temperature Index." February 26, 2004. <http://www.nws.noaa.gov/om/ windchill> (July 31, 2010).

18. Consider the sphere of radius 3 centered at the origin. The plane tangent to the sphere at $(1, 2, 2)$ intersects the x-axis at a point P. Find the coordinates of P.

19. Show that the plane tangent to a sphere at a point P on the sphere is always perpendicular to the vector $\overrightarrow{OP}$ from the center O of the sphere to P. (Hint: Locate the sphere so its center is at the origin in $\mathbf{R}^3$.)

20. The surface $z = 3x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 - 4y^2$ is intersected by the plane $2x - y = 1$. The resulting intersection is a curve on the surface. Find a set of parametric equations for the line tangent to this curve at the point $(1, 1, -\frac{23}{24})$.

21. Consider the cone $z^2 = x^2 + y^2$.

(a) Find an equation of the plane tangent to the cone at the point $(3, -4, 5)$.

(b) Find an equation of the plane tangent to the cone at the point (a, b, c).

(c) Show that every tangent plane to the cone must pass through the origin.

22. Show that the two surfaces

$$S_1\colon z = xy \quad \text{and} \quad S_2\colon z = \tfrac{3}{4}x^2 - y^2$$

intersect perpendicularly at the point $(2, 1, 2)$.

23. Consider the surface $z = x^2 + 4y^2$.

(a) Find an equation for the plane that is tangent to the surface at the point $(1, -1, 5)$.

(b) Now suppose that the surface is intersected with the plane $x = 1$. The resulting intersection is a curve on the surface (and is a curve in the plane $x = 1$ as well). Give a set of parametric equations for the line in $\mathbf{R}^3$ that is tangent to this curve at the point $(1, -1, 5)$. A rough sketch may help your thinking.

24. A turtleneck sweater has been washed and is now tumbling in the dryer, along with the rest of the laundry. At a particular moment t_0, the neck of the sweater measures 18 inches in circumference and 3 inches in length. However, the sweater is 100% cotton, so that at t_0 the heat of the dryer is causing the neck circumference to shrink at a rate of 0.2 in/min, while the twisting and tumbling action is causing the length of the neck to stretch at the rate of 0.1 in/min. How is the volume V of the space inside the neck changing at $t = t_0$? Is V increasing or decreasing at that moment?

25. A factory generates air pollution each day according to the formula

$$P(S, T) = 330S^{2/3}T^{4/5},$$

where S denotes the number of machine stations in operation and T denotes the average daily temperature. At the moment, 75 stations are in regular use and the average daily temperature is 15 °C. If the average temperature is rising at the rate of 0.2 °C/day and the number of stations being used is falling at a rate of 2 per month, at what rate is the amount of pollution changing? (Note: Assume that there are 24 workdays per month.)

26. Economists attempt to quantify how useful or satisfying people find goods or services by means of **utility functions.** Suppose that the utility a particular individual derives from consuming x ounces of soda per week and watching y minutes of television per week is

$$u(x, y) = 1 - e^{-0.001x^2 - 0.00005y^2}.$$

Further suppose that she currently drinks 80 oz of soda per week and watches 240 min of TV each week. If she were to increase her soda consumption by 5 oz/week and cut back on her TV viewing by 15 min/week, is the utility she derives from these changes increasing or decreasing? At what rate?

27. Suppose that $w = x^2 + y^2 + z^2$ and $x = \rho \cos\theta \sin\varphi$, $y = \rho \sin\theta \sin\varphi$, $z = \rho \cos\varphi$. (Note that the equations for x, y, and z in terms of ρ, φ, and θ are just the conversion relations from spherical to rectangular coordinates.)

(a) Use the chain rule to compute $\partial w/\partial\rho$, $\partial w/\partial\varphi$, and $\partial w/\partial\theta$. Simplify your answers as much as possible.

(b) Substitute ρ, φ, and θ for x, y, and z in the original expression for w. Can you explain your answer in part (a)?

28. If $w = f\left(\dfrac{x+y}{xy}\right)$, show that

$$x^2\frac{\partial w}{\partial x} - y^2\frac{\partial w}{\partial y} = 0.$$

(You should assume that f is a differentiable function of one variable.)

29. Let $z = g(x, y)$ be a function of class C^2, and let $x = e^r \cos\theta$, $y = e^r \sin\theta$.

(a) Use the chain rule to find $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$. Use your results to solve for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of $\partial z/\partial r$ and $\partial z/\partial\theta$.

(b) Use part (a) and the product rule to show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial\theta^2}\right).$$

30. (a) Use the function $f(x, y) = x^y\ (= e^{y \ln x})$ and the multivariable chain rule to calculate $\dfrac{d}{du}(u^u)$.

(b) Use the multivariable chain rule to calculate $\dfrac{d}{dt}((\sin t)^{\cos t})$.

31. Use the function $f(x, y, z) = x^{y^z}$ and the multivariable chain rule to calculate $\dfrac{d}{du}\left(u^{u^u}\right)$.

32. Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}$ is a function of class C^2. The **Laplacian** of f, denoted $\nabla^2 f$, is defined to be

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

When $n = 2$ or 3, this construction is important when studying certain differential equations that model physical phenomena, such as the heat or wave equations. (See Exercises 28 and 29 of §2.4.) Now suppose that f depends only on the distance $\mathbf{x} = (x_1, \ldots, x_n)$ is from the origin in $\mathbf{R}^n$; that is, suppose that $f(\mathbf{x}) = g(r)$ for some function g, where $r = \|\mathbf{x}\|$. Show that for all $\mathbf{x} \neq \mathbf{0}$, the Laplacian is given by

$$\nabla^2 f = \frac{n-1}{r} g'(r) + g''(r).$$

33. (a) Consider a function $f(x, y)$ of class C^4. Show that if we apply the Laplacian operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ twice to f, we obtain

$$\nabla^2(\nabla^2 f) = \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}.$$

(b) Now suppose that f is a function of n variables of class C^4. Show that

$$\nabla^2(\nabla^2 f) = \sum_{i,j=1}^{n} \frac{\partial^4 f}{\partial x_i^2 \partial x_j^2}.$$

Functions that satisfy the partial differential equation $\nabla^2(\nabla^2 f) = 0$ are called **biharmonic functions** and arise in the theoretical study of elasticity.

34. Livinia, the housefly, finds herself caught in the oven at the point $(0, 0, 1)$. The temperature at points in the oven is given by the function

$$T(x, y, z) = 10(xe^{-y^2} + ze^{-x^2}),$$

where the units are in degrees Celsius.

(a) If Livinia begins to move toward the point $(2, 3, 1)$, at what rate (in deg/cm) does she find the temperature changing?

(b) In what direction should she move in order to cool off as rapidly as possible?

(c) Suppose that Livinia can fly at a speed of 3 cm/sec. If she moves in the direction of part (b), at what (instantaneous) rate (in deg/sec) will she find the temperature to be changing?

35. Consider the surface given in cylindrical coordinates by the equation $z = r\cos 3\theta$.

(a) Describe this surface in Cartesian coordinates, that is, as $z = f(x, y)$.

(b) Is f continuous at the origin? (Hint: Think cylindrical.)

(c) Find expressions for $\partial f/\partial x$ and $\partial f/\partial y$ at points other than $(0, 0)$. Give values for $\partial f/\partial x$ and $\partial f/\partial y$ at $(0, 0)$ by looking at the partial functions of f through $(x, 0)$ and $(0, y)$ and taking one-variable limits.

(d) Show that the directional derivative $D_{\mathbf{u}} f(0, 0)$ exists for every direction (unit vector) $\mathbf{u}$. (Hint: Think in cylindrical coordinates again and note that you can specify a direction through the origin in the xy-plane by choosing a particular constant value for θ.)

(e) Show directly (by examining the expression for $\partial f/\partial y$ when $(x, y) \neq (0, 0)$ and also using part (c)) that $\partial f/\partial y$ is *not* continuous at $(0, 0)$.

(f) Sketch the graph of the surface, perhaps using a computer to do so.

36. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = c\frac{\partial^2 u}{\partial t^2}$$

is known as the **wave equation**. It models the motion of a wave $u(x, y, z, t)$ in $\mathbf{R}^3$ and was originally derived by Johann Bernoulli in 1727. In this equation, c is a positive constant, the variables x, y, and z represent spatial coordinates, and the variable t represents time.

(a) Let $u = \cos(x - t) + \sin(x + t) - 2e^{z+t} - (y - t)^3$. Show that u satisfies the wave equation with $c = 1$.

(b) More generally, show that if f_1, f_2, g_1, g_2, h_1, and h_2 are any twice differentiable functions of a single variable, then

$$\begin{aligned} u(x, y, z, t) &= f_1(x - t) + f_2(x + t) \\ &\quad + g_1(y - t) + g_2(y + t) \\ &\quad + h_1(z - t) + h_2(z + t) \end{aligned}$$

satisfies the wave equation with $c = 1$.

*Let X be an open set in $\mathbf{R}^n$. A function $F\colon X \to \mathbf{R}$ is said to be **homogeneous of degree** d if, for all $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X$ and all $t \in \mathbf{R}$ such that $t\mathbf{x} \in X$, we have*

$$F(tx_1, tx_2, \ldots, tx_n) = t^d F(x_1, x_2, \ldots, x_n).$$

Exercises 37–44 concern homogeneous functions.

In Exercises 37–41, which of the given functions are homogeneous? For those that are, indicate the degree d of homogeneity.

37. $F(x, y) = x^3 + xy^2 - 6y^3$

38. $F(x, y, z) = x^3 y - x^2 z^2 + z^8$

39. $F(x, y, z) = zy^2 - x^3 + x^2 z$

40. $F(x, y) = e^{y/x}$

41. $F(x, y, z) = \dfrac{x^3 + x^2 y - yz^2}{xyz + 7xz^2}$

42. If $F(x, y, z)$ is a polynomial, characterize what it means to say that F is homogeneous of degree d (i.e., explain what must be true about the polynomial if it is to be homogeneous of degree d).

43. Suppose $F(x_1, x_2, \ldots, x_n)$ is differentiable and homogeneous of degree d. Prove **Euler's formula**:

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} + \cdots + x_n \frac{\partial F}{\partial x_n} = dF.$$

(Hint: Take the equation $F(tx_1, tx_2, \ldots, tx_n) = t^d F(x_1, x_2, \ldots, x_n)$ that defines homogeneity and differentiate with respect to t.)

44. Generalize Euler's formula as follows: If F is of class C^2 and homogeneous of degree d, then

$$\sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} = d(d-1)F.$$

Can you conjecture what an analogous formula involving the kth-order partial derivatives should look like?