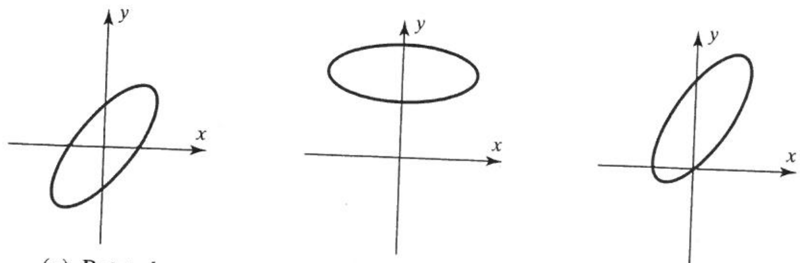

Figure 9.6.2

EXAMPLE 5 Completing the Square and Translating

Since the quadratic equation

$$2x^2 + y^2 - 12x - 4y + 18 = 0$$

contains x^2, x, y^2, and y terms but no cross-product term, its graph is a conic that is translated out of standard position but not rotated. This conic can be brought into standard position by suitably translating coordinate axes. To do this, first collect x and y terms. This yields

$$(2x^2 - 12x) + (y^2 - 4y) + 18 = 0 \quad \text{or} \quad 2(x^2 - 6x) + (y^2 - 4y) = -18$$

By completing the squares† on the two expressions in parentheses, we obtain

$$2(x^2 - 6x + 9) + (y^2 - 4y + 4) = -18 + 18 + 4$$

or

$$2(x-3)^2 + (y-2)^2 = 4 \tag{3}$$

If we translate the coordinate axes by means of the translation equations

$$x' = x - 3, \qquad y' = y - 2$$

then (3) becomes

$$2x'^2 + y'^2 = 4 \quad \text{or} \quad \frac{x'^2}{2} + \frac{y'^2}{4} = 1$$

which is the equation of an ellipse in standard position in the $x'y'$-system. This ellipse is sketched in Figure 9.6.3. ◆

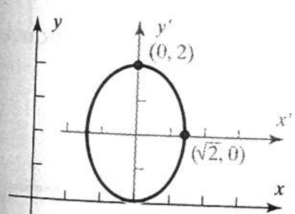

Figure 9.6.3
$\frac{x'^2}{2} + \frac{y'^2}{4} = 1$

Eliminating the Cross-Product Term

We shall now show how to identify conics that are rotated out of standard position. If we omit the brackets on 1×1 matrices, then (2) can be written in the matrix form

$$[x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \quad e]\begin{bmatrix} x \\ y \end{bmatrix} + f = 0$$

or

$$\mathbf{x}^T A\mathbf{x} + K\mathbf{x} + f = 0$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad K = [d \quad e]$$

---

† To complete the square on an expression of the form $x^2 + px$, add and subtract the constant $(p/2)^2$ to obtain

$$x^2 + px = x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

Now consider a conic C whose equation in xy-coordinates is

$$\mathbf{x}^T A\mathbf{x} + K\mathbf{x} + f = 0 \tag{4}$$

We would like to rotate the xy-coordinate axes so that the equation of the conic in the new $x'y'$-coordinate system has no cross-product term. This can be done as follows.

Step 1. Find a matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

that orthogonally diagonalizes the matrix A.

Step 2. Interchange the columns of P, if necessary, to make $\det(P) = 1$. This assures that the orthogonal coordinate transformation

$$\mathbf{x} = P\mathbf{x}', \quad \text{that is,} \quad \begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x' \\ y' \end{bmatrix} \tag{5}$$

is a rotation.

Step 3. To obtain the equation for C in the $x'y'$-system, substitute (5) into (4). This yields

$$(P\mathbf{x}')^T A(P\mathbf{x}') + K(P\mathbf{x}') + f = 0$$

or

$$(\mathbf{x}')^T(P^T AP)\mathbf{x}' + (KP)\mathbf{x}' + f = 0 \tag{6}$$

Since P orthogonally diagonalizes A,

$$P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are eigenvalues of A. Thus, (6) can be rewritten as

$$\begin{bmatrix} x' & y' \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + f = 0$$

or

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f = 0$$

(where $d' = dp_{11} + ep_{21}$ and $e' = dp_{12} + ep_{22}$). This equation has no cross-product term.

The following theorem summarizes this discussion.

Theorem 9.6.2 Principal Axes Theorem for R^2

Let

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

be the equation of a conic C, and let

$$\mathbf{x}^T A\mathbf{x} = ax^2 + 2bxy + cy^2$$

be the associated quadratic form. Then the coordinate axes can be rotated so that the equation for C in the new $x'y'$-coordinate system has the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f = 0$$

where λ_1 and λ_2 are the eigenvalues of A. The rotation can be accomplished by the substitution

$$\mathbf{x} = P\mathbf{x}'$$

where P orthogonally diagonalizes A and $\det(P) = 1$.

EXAMPLE 6 Eliminating the Cross-Product Term

Describe the conic C whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$.

Solution.

The matrix form of this equation is

$$\mathbf{x}^T A\mathbf{x} - 36 = 0 \tag{7}$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic equation of A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{bmatrix} = (\lambda - 9)(\lambda - 4) = 0$$

so the eigenvalues of A are $\lambda = 4$ and $\lambda = 9$. We leave it for the reader to show that orthonormal bases for the eigenspaces are

$$\lambda = 4: \quad \mathbf{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \qquad \lambda = 9: \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Thus,

$$P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

orthogonally diagonalizes A. Moreover, $\det(P) = 1$ so that the orthogonal coordinate transformation

$$\mathbf{x} = P\mathbf{x}' \tag{8}$$

is a rotation. Substituting (8) into (7) yields

$$(P\mathbf{x}')^T A(P\mathbf{x}') - 36 = 0 \quad \text{or} \quad (\mathbf{x}')^T(P^T AP)\mathbf{x}' - 36 = 0$$

Since

$$P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

this equation can be written as

$$[x' \quad y']\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - 36 = 0$$

or

$$4x'^2 + 9y'^2 - 36 = 0 \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

which is the equation of the ellipse sketched in Figure 9.6.4. In that figure, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are the column vectors of P. ♦

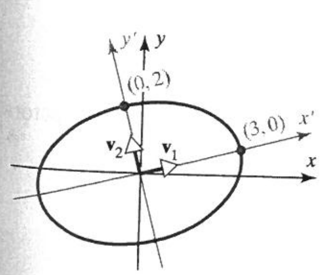

Figure 9.6.4 $\frac{x'^2}{9} + \frac{y'^2}{4} = 1$

EXAMPLE 7 Eliminating the Cross-Product Term Plus Translation

Describe the conic C whose equation is

$$5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0$$

Solution.

The matrix form of this equation is

$$\mathbf{x}^T A\mathbf{x} + K\mathbf{x} + 4 = 0 \tag{9}$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \quad \text{and} \quad K = [4\sqrt{5} \;\; -16\sqrt{5}\,]$$

As shown in Example 6,

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

orthogonally diagonalizes A and has determinant 1. Substituting $\mathbf{x} = P\mathbf{x}'$ into (9) gives

$$(P\mathbf{x}')^T A(P\mathbf{x}') + K(P\mathbf{x}') + 4 = 0$$

or

$$(\mathbf{x}')^T(P^T AP)\mathbf{x}' + (KP)\mathbf{x}' + 4 = 0 \tag{10}$$

Since

$$P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \quad \text{and} \quad KP = \begin{bmatrix} \frac{20}{\sqrt{5}} & -\frac{80}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = [-8 \;\; -36]$$

(10) can be written as

$$4x'^2 + 9y'^2 - 8x' - 36y' + 4 = 0 \tag{11}$$

To bring the conic into standard position, the $x'y'$ axes must be translated. Proceeding as in Example 5, we rewrite (11) as

$$4(x'^2 - 2x') + 9(y'^2 - 4y') = -4$$

Completing the squares yields

$$4(x'^2 - 2x' + 1) + 9(y'^2 - 4y' + 4) = -4 + 4 + 36$$

or

$$4(x' - 1)^2 + 9(y' - 2)^2 = 36 \tag{12}$$

If we translate the coordinate axes by means of the translation equations

$$x'' = x' - 1, \qquad y'' = y' - 2$$

then (12) becomes

$$4x''^2 + 9y''^2 = 36 \quad \text{or} \quad \frac{x''^2}{9} + \frac{y''^2}{4} = 1$$

which is the equation of the ellipse sketched in Figure 9.6.5. In that figure, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are the column vectors of P. ♦

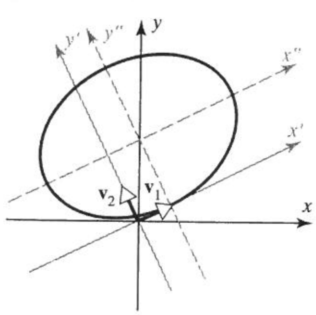

Figure 9.6.5
$\frac{x''^2}{9} + \frac{y''^2}{4} = 1$

Exercise Set 9.6

1. In each part find a change of variables that reduces the quadratic form to a sum or difference of squares, and express the quadratic form in terms of the new variables.

   (a) $2x_1^2 + 2x_2^2 - 2x_1x_2$ (b) $5x_1^2 + 2x_2^2 + 4x_1x_2$ (c) $2x_1x_2$ (d) $-3x_1^2 + 5x_2^2 + 2x_1x_2$