MAT 258 Final Exam Answer Sheet

Summer 2021

Quiz ID: JST Name: ____________________

1. ☐	11. ☐	21. ☐
2. ☐	12. ☐	22. ☐
3. ☐	13. ☐	23. ☐
4. ☐	14. ☐	24. ☐
5. ☐	15. ☐	25. ☐
6. ☐	16. ☐	26. ☐
7. ☐	17. ☐	27. ☐
8. ☐	18. ☐	28. ☐
9. ☐	19. ☐	29. ☐
10. ☐	20. ☐	30. ☐

Do your own work, and submit electronic answers at

http://azrael.digipen.edu/cgi-bin/MAT258quiz.pl

and submit scratch work on Moodle.

Answers and scratch work due by midnight on Wednesday, July 21.

No late work or answers will be accepted.

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MAT 258 **Final Exam** Summer 2021

1. How many bit strings are there of length 10 with exactly 3 ones or exactly 4 zeros?

   a) 480 b) 320 c) $(4+3)^{10}$ d) 330 e) $10^3 + 10^4$

2. How many bit strings are there of length 10 which begin with 00 or end with 11?

   a) $2^9 - 2^6$ b) 320 c) $2^{10} - 10^2$ d) 440 e) $2^8 - 2^6$

3. How many bit strings of length 8 have exactly 2 zeros, and every occurence of a zero is followed by a one, somewhere else in the sequence, but not necessarily immediately?

   a) 24 b) 21 c) 15 d) 36 e) 16

4. Let a_i be the number of subsets of size i of a set of size n. Evaluate the following sum using the binomial theorem where n is a positive integer:

   $$a_0 + a_1 + a_2 + \cdots + a_n.$$

   a) 2^{n-1} b) 2^n c) $n \cdot 2^n$ d) $n!$ e) n

5. Evaluate the alternating sum (using the binomial theorem) where n is an *odd* integer:

   $$\binom{n}{0}2^n - \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} - \binom{n}{3}2^{n-3} + - \cdots + \binom{n}{n-1}2 - \binom{n}{n}.$$

   a) 2^{n-1} b) 2^n c) -1 d) 0 e) 1

6. Let k and r be integers with $1 \leq k \leq r$. Let X be a set with r elements, let $x \in X$, and let $Y = X \backslash \{x\}$ (ie. all elements of X except x.) Let s be the number of subsets of Y of size $k-1$, and let t be number of subsets of Y of size k. Which of the following is equivalent to $s + t$?

   a) $\binom{r}{k}$ b) $\binom{r+1}{k+1}$ c) 2^{r-1} d) 2^k e) 1

7. How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 10$ have with integers $x_i \geq 0$?

   a) 262 b) 380 c) 308 d) 250 e) 286

8. How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 10$ have with integers $x_i \geq 0$ with the additional restrictions: $x_1 \leq 3$ and $x_2 \leq 4$?

   a) 175 b) 170 c) 145 d) 150 e) 140

9. How many strings containing only digits 0,1, or 2, contain exactly two 0's, three 1's, and four 2's ?

   a) 1440 b) 1320 c) 1260 d) 1184 e) 1024

10. In lexicographic order, how many permutations follow 45231 in the list of all permuations of the numbers 1,2,3,4,5 ?

    a) 54 b) 26 c) 32 d) 42 e) 20

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11. Use the Euclidean Algorithm to find the greatest common divisor d, or GCD, of the integers 1525 and 390, and also to find integers m and n so that $m \cdot 1525 + n \cdot 390 = d$. What is $m + n$?

a) -33 b) -32 c) -31 d) -34 e) -30

12. How many different alphabetized (unlabelled) strings using one or two blanks in place of one or two letters can be formed from the word VACCINE? (Two blanks can be used to replace two of the same letter or two different letters.)

a) 25 b) 21 c) 24 d) 23 e) 22

13. How many different draws from a scrabble bag will result in the ability to spell the word VACCINE assuming that none of the tiles is a blank? (Take into account the frequency of each letter, as in the file scrabble-bag.txt on the website.)

a) 15664 b) 11664 c) 13664 d) 14664 e) 12664

14. If two fair dice are rolled, let x be the number on die number 1 and y be the number on die number 2, and (x, y) be a point in the plane with distance $D = \sqrt{x^2 + y^2}$ from the origin. What is the probability that $D < 5$?

a) $\frac{7}{12}$ b) $\frac{5}{12}$ c) $\frac{13}{36}$ d) $\frac{1}{3}$ e) $\frac{7}{18}$

15. Same x, y, and D as in the previous problem. What is the probability that the distance D will be an integer?

a) $\frac{1}{9}$ b) $\frac{1}{12}$ c) $\frac{1}{36}$ d) $\frac{1}{6}$ e) $\frac{1}{18}$

16. Consider the proposition X given by $(p \to q) \to (q \to r)$. Which of the following are necessary for X to be False?

i) p is False ii) q is True iii) r is False

a) i) only b) ii) only c) ii) and iii) only d) i) and iii) only e) i) and ii) only

17. Same proposition X as in the previous problem. Suppose a bit string of length three is generated at random, and the values of the bit string are interpreted as T/F values for three propositions p, q and r, in that order. What is the probability that the proposition X is True?

a) $\frac{3}{8}$ b) $\frac{1}{4}$ c) $\frac{1}{8}$ d) $\frac{7}{8}$ e) $\frac{3}{4}$

18. It is a fact that any prime number p is equal to the sum of the squares of two positive integers if and only if $p \equiv 1 \ (mod\ 4)$. Choose a logical statement of this fact, given the following: Let S be the set of primes, and let $r_4(k)$ be the remainder when a positive integer k is divided by 4, so that $k = q \cdot 4 + r_4(k)$. Also let X be the statement: $r_4(p) = 1$, and let Y be the statement: $p = m^2 + n^2$. (Unless otherwise specified, assume that quantifiers range over the positive integers.)

a) $X \longleftrightarrow Y$ b) $\forall p \in S : \exists m, n : (X \longleftrightarrow Y)$ c) $\exists m, n : (X \longleftrightarrow \forall p \in S : Y)$ d) $\exists m, n : \forall p \in S : (X \longleftrightarrow Y)$ e) $\forall p \in S : (X \longleftrightarrow \exists m, n : Y)$

19. Let S be the set of primes. A squarefree positive integer n is one which is not divisible by p^2 for any prime number $p \in S$. Equivalently, the prime factorization of n is a product of prime powers $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ with all exponents $e_i = 1$ for $i = 1, \ldots, k$. Choose a logical statement that is equivalent to the *negation* of the claim: "All positive integers are squarefree". (Unless otherwise specified, assume that quantifiers range over the positive integers.)

a) $\exists n : \forall p \in S : p^2 | n$ b) $\forall p \in S : \forall n : p^2 | n$ c) $\forall p \in S : \exists n : p^2 | n$ d) $\exists p \in S : \forall n : p^2 | n$ e) $\exists n : \exists p \in S : p^2 | n$

20. Let X be the statement that a composite positive integer n is equal to a sum of two squares of positive integers. Let Y be the statement that a positive integer n is squarefree. Let Z be the statement that all prime factors of a positive integer n are congruent to 1 modulo 4. Then for X to be True it is a sufficient condition to know that both Y and Z are True. Choose a logical proposition that expresses this fact correctly.

a) $\forall n : (X \wedge Z) \longrightarrow Y$ b) $\forall n : (X \wedge Y) \longrightarrow Z$ c) $\forall n : Z \longrightarrow (X \wedge Y)$ d) $\forall n : X \longrightarrow (Y \wedge Z)$ e) $\forall n : (Y \wedge Z) \longrightarrow X$

21. Same statements X and Y as in the previous question. Let W be the statement that at least one prime factor of a positive integer n is congruent to 3 modulo 4. Then for X to be False it is a sufficient condition to know that both Y and W are True. Choose a logical proposition that expresses this fact correctly.

a) $\forall n : (\neg X \wedge W) \longrightarrow Y$ b) $\forall n : (X \wedge Y) \longrightarrow \neg W$ c) $\forall n : \neg W \longrightarrow (X \wedge Y)$ d) $\forall n : \neg X \longrightarrow (Y \wedge W)$ e) $\forall n : (Y \wedge W) \longrightarrow \neg X$

22. Based on the previous two questions, which of the following integers can be expressed as the sum of two squares of positive integers? (To find at least one prime factor just check primes less than 20.)

a) 247 b) 437 c) 1463 d) 187 e) 1105

23. Choose the description that best fits the following statement and observation:

Claim: A positive integer which is greater than 5 and is one more than a perfect square, cannot be a prime number. Observation: Let $m = 17 = 4^2 + 1$.

a) disproof by counterexample b) proof by contradiction c) incorrect proof d) indirect proof e) direct proof

24. Suppose G is the domineering game below, where X indicates that a square cannot be used.

		X

Assuming both players play randomly, so that any move is equally likely, find the probabilities: P(Lwpf) and Pi(Lwps). What is the sum P(Lwpf) + P(Lwps)?

a) 1 b) $\frac{4}{3}$ c) $\frac{3}{2}$ d) $\frac{3}{4}$ e) $\frac{5}{6}$

25. Same Domineering game G as in the previous question. If we roll a fair die to see who plays first, with Left playing first if a 1 is rolled, and Right playing first for all other numbers rolled, and then players play randomly, what is the probability that Left will win?

a) $\frac{17}{18}$ b) $\frac{7}{18}$ c) $\frac{23}{36}$ d) $\frac{13}{36}$ e) $\frac{7}{12}$

26. Assume that a property X of games is verified to be true for games of height zero. It is then also shown that if property X is assumed to be true for all games of height at most $n-1$, then property X must be true for all games of height n. Now suppose that G is a particular game of height 12. In addition to the above, what do we need to do in order to prove that G has property X?

a) verify property X for $n \leq 12$ b) verify property X for $n \leq 11$ c) verify property X for $n = 11$ d) verify property X for $n = 12$ e) nothing

27. Let Y be the property that a game is impartial. (A game is impartial if the options are the same for either player at any node of the game tree.) Which of the following fails to be true, or fails to be true by assumption, or fails to be provable, in attempting to prove by induction on game tree height that all games are impartial?

i) the base case ii) the induction step iii) the induction hypothesis

a) none of the above b) i) and ii) only c) iii) only d) ii) only e) i) only

28. Which of the following compound statements always evaluate to FALSE for any combinatorial game G? (Recall that Lwpf reads "Left has a winning strategy playing first", etc.)

i) Lwpf and Rwps ii) Lwpf and Rwpf iii) Lwps and Rwps

a) none of the above b) i) and ii) only c) iii) only d) ii) only e) i) only

29. Let Y be the property that a game is in Lwpf or Rwps. (Recall that Lwpf also refers to the set of all games in which Left has a winning strategy playing first, etc.) Which of the following fails to be true, or fails to be true by assumption, or fails to be provable, in attempting to prove by induction on game tree height that all games are in Lwpf or Rwps?

i) the base case ii) the induction step iii) the induction hypothesis

a) none of the above b) i) and ii) only c) iii) only d) ii) only e) i) only

30. Use the Chinese Remainder Theorem to solve the system of congruences for x:

$$x \equiv 5 \ (mod\ 6), \quad x \equiv 3 \ (mod\ 5), \quad x \equiv 2 \ (mod\ 11)$$

Let x_1 be the unique solution $(mod\ 330)$. Now modify the system and solve again for x:

$$x \equiv 5 \ (mod\ 6), \quad x \equiv 3 \ (mod\ 5), \quad x \equiv 3 \ (mod\ 11)$$

Let x_2 be the unique solution $(mod\ 330)$. What is $x_1 - x_2$?

a) 2! b) 6! c) 4! d) 5! e) 3!