Confluent Vandermonde Determinants

May 12, 2010

We associate to a string

$$s = a_1^{e_1} \cdots a_n^{e_n} = b_0 \cdots b_d, \quad \sum_{i=1}^{n} e_i = d+1,$$

the confluent Vandermonde determinant $D(s)$, which has e_i rows involving a_i for each i:

$$D(s) = \begin{vmatrix} 1 & a_1 & a_1^2 & a_1^3 & \dots & a_1^d \\ 0 & 1 & 2a_1 & 3a_1^2 & \dots & da_1^{d-1} \\ 0 & 0 & 2 & 6a_1 & \dots & d(d-1)a_1^{d-2} \\ & & & & \vdots & \\ 1 & a_2 & a_2^2 & a_2^3 & \dots & a_2^d \\ 0 & 1 & 2a_2 & 3a_2^2 & \dots & da_2^{d-1} \\ 0 & 0 & 2 & 6a_2 & \dots & d(d-1)a_2^{d-2} \\ & & & & \vdots & \\ 1 & a_n & a_n^2 & a_n^3 & \dots & a_n^d \\ 0 & 1 & 2a_n & 3a_n^2 & \dots & da_n^{d-1} \\ 0 & 0 & 2 & 6a_n & \dots & d(d-1)a_n^{d-2} \\ & & & & \vdots & \end{vmatrix}.$$

Let $P(s)$ be the product of all backward differences of the b_i:

$$P(s) = \prod_{i<j} b_j - b_i.$$

In the case where all $e_i = 1$, we have the Van der Monde determinant formula $D(s) = P(s)$. In the case where some $e_i > 1$, we need a modified product, which involves throwing out all of the zero factors, and also introducing some factorials for the repeated values:

For each repeated string a^k let $k!! = k!(k-1)! \cdots 2!1!$, and let $\Pi(s)$ be the product:

$$\Pi(s) = \left(\prod_{\substack{i<j \\ b_i \neq b_j}} b_j - b_i \right) \prod_{k=1}^{n} (e_k - 1)!!.$$

Theorem: $D(s) = \Pi(s)$.

Proof: Let s be a string of real numbers as above, and let $D(s)$ be the associated determinant. Let $K(s)$ be the maximum of the e_i, $i = 1, \dots, n$. If $K(s) = 1$, then we are in the Van der Monde case, and $D(s) = P(s) = \Pi(s)$. Let $J(s)$ be the sum $J(s) = \sum_{i=1}^{n}(e_i - 1)$. Then $K(s) = 1$ if and only if $J(s) = 0$. We will proceed by induction on J. We will assume that the theorem is true for all strings r with $K(r) = K(s) - 1$.

Let $e_m = K(s)$ for some index m, and let r be the string which is indentical to s except that the last occurence of a_m is replaced by x. So r looks like:

$$\begin{aligned} r &= a_1^{e_1} \cdots a_m^{e_m-1} x \cdots a_n^{e_n} \\ &= b_0 \cdots b_{\kappa-1} x b_{\kappa+1} \cdots b_d. \end{aligned}$$

When we write r using the b_i, then r is identical to s except that $b_\kappa = x$ for some index κ. Note that $J(r) = J(s) - 1$, although it is possible that $K(r) = K(s)$. Then $D(r) = p(x)$ is a polynomial in x of degree at most d. We call this the "determinant form" of $p(x)$. The derivatives of $p(x)$ can be obtained by differentiating the row of the matrix containing x, and taking the new determinant. By repeated differentiation, we can see that $D(s)$ is simply a derivative of p evaluated at a_m:

$$D(s) = p^{(e_m-1)}(a_m).$$

By induction, we can also assume that $p(x)$ has the polynomial formula:

$$\begin{aligned} p(x) &= \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k \neq m} (e_k - 1)!! \right) (e_m - 2)!! \prod_{l=0}^{\kappa-1} x - b_l \prod_{t=\kappa+1}^{d} b_t - x \\ &= \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k \neq m} (e_k - 1)!! \right) (e_m - 2)!! \cdot (x - a_m)^{e_m - 1} \cdot q_1(x). \end{aligned}$$

where $q_1(a_m) \neq 0$. Differentiating, we can write:

$$p^{(e_m-1)}(x) = \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k \neq m} (e_k - 1)!! \right) (e_m - 2)!! \left[(e_m - 1)! \cdot q_1(x) + q_2(x) \right]$$

where $q_2(a_m) = 0$. Then, noting that $a_m = b_\kappa$:

$$\begin{aligned} D(s) &= p^{(e_m-1)}(a_m) \\ &= \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k \neq m} (e_k - 1)!! \right) (e_m - 2)!!(e_m - 1)! q_1(a_m) \\ &= \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k \neq m} (e_k - 1)!! \right) (e_m - 1)!! \prod_{\substack{l=0 \\ b_l \neq a_m}}^{\kappa-1} a_m - b_l \prod_{\substack{t=\kappa+1 \\ b_t \neq a_m}}^{d} b_t - a_m \\ &= \left(\prod_{\substack{i<j, b_i \neq b_j \\ i \neq \kappa \neq j}} b_j - b_i \right) \left(\prod_{k=1}^{n} (e_k - 1)!! \right) \prod_{l<\kappa} b_\kappa - b_l \prod_{\kappa+1<t} b_t - b_\kappa \\ &= \left(\prod_{\substack{i<j \\ b_i \neq b_j}} b_j - b_i \right) \prod_{k=1}^{n} (e_k - 1)!! \\ &= \Pi(s). \end{aligned}$$