

MAT 300/500 Homework 3 – Spring 2025

Due Date: Tuesday, Feb 11

The *Cumulative form* (CB-form) of a Bezier Curve $\gamma(t) = \sum_{i=0}^d P_i B_i^d(t)$ is:

$$\gamma(t) = P_0 + \sum_{i=1}^d (P_i - P_{i-1}) C_i^d(t) = P_0 + \sum_{i=1}^d C_i^d(t) \mathbf{v}_i.$$

The differences of points $P_i - P_{i-1}$ can be interpreted as vectors \mathbf{v}_i , and P_0 can also be interpreted as the endpoint of a vector. The derivative of a Bezier Curve can be obtained from this form and is:

$$\gamma'(t) = d \sum_{i=0}^{d-1} (P_{i+1} - P_i) B_i^{d-1}(t) = d \sum_{i=0}^{d-1} B_i^{d-1}(t) \mathbf{v}_{i+1}.$$

It may be helpful to draw some examples and note the role played by the vectors \mathbf{v}_i for some quadratic Bezier curves before starting problem one. In most cases, for quadratic Bezier curves, the two vectors \mathbf{v}_1 and \mathbf{v}_2 will be linearly independent. When they are linearly dependent, we call this a degenerate case, which is what happens in problem one.

1. (a) Find the Cumulative form of the quadratic Bezier curve $\gamma(t)$ with control points $P_0 = (1, 0)$, $P_1 = (2, 1)$, and $P_2 = (4, 3)$. (Write it as a sum of polynomials and vectors based on the control points.)
 (b) Write the derivative $\gamma'(t)$ as a parametric polynomial curve.
 (c) Find a value of t when the derivative $\gamma'(t) = 0$.
 (d) Find an implicit equation (involving x and y but not t) for the set S of points (x, y) in the plane such that $\gamma(t) = (x(t), y(t))$, which is true for all t . (Hint: Just find the simplest x, y equation through the points.)
 (e) How many times does each point defined by this implicit equation occur as a value of $\gamma(t)$? (Hint: Some points occur twice or not at all, and one point occurs only once.)
 (f) Find a new parametrization (not quadratic) that covers the same set S of points exactly once. (ie. give an interval of t -values so that there is a 1-1 correspondence between those t -values and the points in S .)
2. This exercise explores the idea of approximating a circular path with Bezier curves. Let $f(t)$ be the arc-length parametrization of the unit circle from the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, normalized so that the parameter t goes from 0 to 1:

$$f(t) = \left(\cos \frac{\pi}{2} \left(t - \frac{1}{2} \right), \sin \frac{\pi}{2} \left(t - \frac{1}{2} \right) \right).$$

- (a) Sketch a picture of the curve in $f(t)$ together with its tangent vectors at the points where $t = 0$, $t = \frac{1}{2}$, and $t = 1$.
- (b) Find a quadratic Bezier curve $\gamma(t)$ which has the same values as f at $t = 0$ and at $t = 1$: $f(0) = \gamma(0)$ and $f(1) = \gamma(1)$, and such that the tangent vectors $\gamma'(0)$ and $\gamma'(1)$ are constant multiples of $f'(0)$ and $f'(1)$ respectively. (Hint: Once you have the control points, you have the Bezier curve, so just pick the control points carefully so that all of the above requirements are satisfied.)
- (c) Find the difference: $\gamma(\frac{1}{2}) - f(\frac{1}{2})$. Find the difference: $\gamma'(\frac{1}{2}) - f'(\frac{1}{2})$.
- (d) Find a cubic Bezier curve $\alpha(t)$ which has the same values as f at $t = 0$, $t = \frac{1}{2}$, and at $t = 1$, and such that the tangent vectors $\alpha'(0)$, $\alpha'(\frac{1}{2})$ and $\alpha'(1)$ are constant multiples of $f'(0)$, $f'(\frac{1}{2})$ and $f'(1)$ respectively. (Hint: Assume that the two control points which are not on the curve $\alpha(t)$ lie on the line $x = a$, with $1 < a < \sqrt{2}$. Then use the conditions above to solve for a .)
- (e) Write the x -coordinate of $\alpha(t)$ as a polynomial in t with coefficients involving $\sqrt{2}$. Simplify, showing that $x(t)$ is in fact a quadratic polynomial.