MAT 300/500 Homework 4 – Fall 2021

Due Date: Tuesday, Feb 23

The *Cumulative form* (CB-form) of a Bezier Curve $\gamma(t) = \sum_{i=0}^{d} P_i B_i^d(t)$ is:

$$\gamma(t) = P_0 + \sum_{i=1}^{d} (P_i - P_{i-1}) C_i^d(t) = P_0 + \sum_{i=1}^{d} C_i^d(t)\mathbf{v}_i.$$

The differences of points $P_i - P_{i-1}$ can be interpreted as vectors $\mathbf{v}_i$, and P_0 can also be interpreted as the endpoint of a vector. The derivative of a Bezier Curve can be obtained from this form and is:

$$\gamma'(t) = d\sum_{i=0}^{d-1} (P_{i+1} - P_i) B_i^{d-1}(t) = d\sum_{i=0}^{d-1} B_i^{d-1}(t)\mathbf{v}_{i+1}.$$

It may be helpful to draw some examples and note the role played by the vectors $\mathbf{v}_i$ for some quadratic Bezier curves before starting problem one. In most cases, for quadratic Bezier curves, the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ will be linearly independent. When they are linearly dependent, we call this is a degenerate case, which is what happens in problem one.

1. (a) Find the Cumulative form of the quadratic Bezier curve $\gamma(t)$ with control points $P_0 = (1, 0)$, $P_1 = (2, 1)$, and $P_2 = (4, 3)$. (Write it as a sum of polynomials and vectors based on the control points.)
   (b) Write the derivative $\gamma'(t)$ as a parametric polynomial curve.
   (c) Find a value of t when the derivative $\gamma'(t) = 0$.
   (d) Find an implicit equation (involving x and y but not t) for the set S of points (x, y) in the plane such that $\gamma(t) = (x(t), y(t))$, which is true for all t. (Hint: Just find the simplest x, y equation through the points.)
   (e) How many times does each point defined by this implicit equation occur as a value of $\gamma(t)$? (Hint: Some points occur twice or not at all, and one point occurs only once.)
   (f) Find a new parametrization (not quadratic) that covers the same set S of points exactly once. (ie. give an interval of t-values so that there is a 1-1 correspondence between those t-values and the points in S.)

2. This exercise explores the idea of approximating a circular path with Bezier curves. Let $f(t)$ be the arc-length parametrization of the unit circle from the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, normalized so that the parameter t goes from 0 to 1:
$$f(t) = (\cos\frac{\pi}{2}(t - \frac{1}{2}), \sin\frac{\pi}{2}(t - \frac{1}{2})).$$
   (a) Sketch a picture of the curve in $f(t)$ together with its tangent vectors at the points where $t = 0$, $t = \frac{1}{2}$, and $t = 1$.
   (b) Find a quadratic Bezier curve $\gamma(t)$ which has the same values as f at $t = 0$ and at $t = 1$: $f(0) = \gamma(0)$ and $f(1) = \gamma(1)$, and such that the tangent vectors $\gamma'(0)$ and $\gamma'(1)$ are constant multiples of $f'(0)$ and $f'(1)$ respectively. (Hint: Once you have the control points, you have the Bezier curve, so just pick the control points carefully so that all of the above requirements are satisfied.)
   (c) Find the difference: $\gamma(\frac{1}{2}) - f(\frac{1}{2})$. Find the difference: $\gamma'(\frac{1}{2}) - f'(\frac{1}{2})$.
   (d) Find a cubic Bezier curve $\alpha(t)$ which has the same values as f at $t = 0$ $t = \frac{1}{2}$, and at $t = 1$, and such that the tangent vectors $\alpha'(0)$, $\alpha'(\frac{1}{2})$ and $\alpha'(1)$ are constant multiples of $f'(0)$, $f'(\frac{1}{2})$ and $f'(1)$ respectively. (Hint: Assume that the two control points which are not on the curve $\alpha(t)$ lie on the line $x = a$, with $1 < a < \sqrt{2}$. Then use the conditions above to solve for a.)
   (e) Write the x-coordinate of $\alpha(t)$ as a polynomial in t with coefficients involving $\sqrt{2}$. Simplify, showing that $x(t)$ is in fact a quadratic polynomial.