MAT 300 Homework 6 – Spring 2021

Due Date: Tuesday, March 23

1. Let V be the vector space $P^4_{1,0}([0,1,2,3,4])$ of continuous linear splines on the indicated sequence of intervals. Let f be the spline in V whose graph is the piecewise linear path connecting the points: $(0,1)$, $(2,-1)$, $(3,1)$, and $(4,1)$ (and consisting of exactly one line between each pair of points.) Let

$$F = \{1, t, (t-1)_+, (t-2)_+, (t-3)_+\}$$

(as functions restricted to $[0,4]$) be a basis of V.

Let h be the 'hat function':

$$h(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases},$$

and let H be the set of functions

$$\{h_1(t) = h(t+1), h_2(t) = h(t), h_3(t) = h(t-1), h_4(t) = h(t-2), h_5(t) = h(t-3)\}$$

restricted to the interval $[0,4]$. Then H is also a basis of V.

   (a) Find the coordinate vector of f with respect to the basis F.
   (b) Find the change of basis matrices from H to F and F to H.
   (c) Find the coordinate vector of f with respect to the basis H.

2. Show that $P^2_{d,0}([a,c,b])$ is a vector subspace of $P^2_d([a,c,b])$. (Quote a criterion from Linear Algebra to check whether a subset is a subspace. Then use a fact from Calculus to verify that you have a subspace.)

3. Considered as subspaces of $C^0([0,3])$, the space of all continuous functions on the interval $[0,3]$, what is the intersection of the vector spaces $P^2_{3,0}([0,1,3])$ and $P^2_{3,0}([0,2,3])$? Give reasons for your answer, without reference to any bases.

4. For each of the following vector spaces give two knot sequences and their corresponding bases of the given vector space: i) a truncated power basis, and ii) a B-spline basis. (Note: write the truncated power functions explicitly based on their definition, but only write the B-splines with their indexed symbols.)

   (a) $P^5_{3,1}[0,1,2,3,4,5]$
   (b) $P^5_{3,\mathbf{r}}[0,1,2,3,4,5]$, $\mathbf{r} = (1,0,-1,2)$
   (c) $P^3_{5,2}[1,2,3,5]$
   (d) $P^3_{5,\mathbf{r}}[1,2,3,5]$, $\mathbf{r} = (0,3)$
   (e) $P^4_{6,5}[0,1,2,3,4]$

5. This problem constructs a quadratic spline (smooth hat function) with divided differences.

   (a) Show that if $c < 0$ is a constant, $g(x) = (c-x)^2_+$ is a function of x, then $[0,1,2,3]g = 0$. (Use the divided difference table.)
   (b) Show that if $c > 3$ is a constant, $g(x) = (c-x)^2_+$ is a function of x, then $[0,1,2,3]g = 0$. (Use the definition, not the table.)
   (c) Let c be in one of the intervals $[0,1)$, $[1,2)$, or $[2,3]$. In each of these three cases, work out the divided difference table to compute $[0,1,2,3](c-x)^2_+$ treating c as a constant.
   (d) Graph the function $f(t) = -3[0,1,2,3](t-x)^2_+$ for all real numbers t. Note: the divided difference is computed with t as a constant and with the data function $g(x) = (t-x)^2_+$ as a function of x.