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MAT 300/500 **Midterm Exam** Spring 2021

Note: For True-False questions, a statement is only True if it must always be True under the given assumptions, otherwise it is False.

1. What is the dimension of the vector subspace of P_2 spanned by the polynomials $(1-t)^2$, $2t(1-t)$, and $1-t^2$?

   a) 5 b) 1 c) 4 d) 2 e) 3

   Correct Answer: 2

2. A linear combination of polynomials $p_0(t), \ldots, p_n(t)$ is a sum $a_0p_0(t)+\cdots+a_np_n(t)$, with real number coefficients a_i. The derivative of a Bernstein polynomial of degree d can be written as a linear combination of:

   i) polynomials $1, t, t^2, \ldots, t^{d-1}$

   ii) two Bernstein polynomials of degree $d-1$

   iii) two polynomials from the standard basis of P_{d-1}

   True or False (answers are in the order i),ii),iii))

   a) FTF b) TTT c) TFT d) FFF e) TTF

   Correct Answer: TTF

3. Let $P_0 = (1, 2)$, $P_1 = (0, -1)$, and $P_2 = (-1, 1)$, and let $\gamma(t)$ be the Bezier curve with control points P_0, P_1, and P_2. Find the Bezier point P_0^1 which is used to compute $\gamma(\frac{1}{2})$. (Sketch the NLI steps.)

   a) $(\frac{1}{8}, \frac{3}{4})$ b) $(\frac{3}{4}, \frac{1}{4})$ c) $(\frac{1}{4}, \frac{1}{2})$ d) $(\frac{1}{2}, \frac{1}{2})$ e) $(\frac{3}{2}, \frac{1}{4})$

   Correct Answer: $(\frac{1}{2}, \frac{1}{2})$

4. Same $\gamma(t)$ as in the previous question. Find $\gamma(\frac{1}{2})$.

   a) $(0, \frac{1}{4})$ b) $(-\frac{1}{2}, 0)$ c) $(\frac{1}{8}, \frac{1}{8})$ d) $(\frac{3}{4}, 0)$ e) $(\frac{1}{4}, \frac{1}{4})$

   Correct Answer: $(0, \frac{1}{4})$

5. Suppose $g(1) = 3$, $g(2) = 1$, and $g(3) = 1$. Find the interpolating polynomial $p(t)$ in P_2 which matches this data. What is $p'(3)$?

   a) 2 b) 0 c) 3 d) -2 e) 1

   Correct Answer: 1

6. Suppose $g(1) = 3$, $g(2) = 1$, $g(3) = 1$, and also $g'(3) = 2$. Find the interpolating polynomial $p(t)$ in P_3 which matches this data. What is the coefficient of t^3 in this $p(t)$?

   a) -2 b) 1 c) $\frac{1}{2}$ d) $\frac{3}{2}$ e) $-\frac{1}{2}$

   Correct Answer: $\frac{1}{2}$

7. Recall that a function f has a zero of multiplicity $r+1$ at $t=c$ if $f^{(k)}(c)=0$ for $k=0,\ldots,r$. We say that f has a zero of *exact* multiplicity $r+1$ if additionally $f^{(r+1)}(c)\neq 0$. Suppose the Bernstein polynomial $B_i^d(t)$ has a zero of exact multiplicity 3 at $t=0$ and a zero of exact multiplicity 3 at $t=1$. What is the coefficient of t^6 in this polynomial?

   a) 20 b) -20 c) 10 d) -10 e) 6

   Correct Answer: -20

8. Let $\gamma(t)=\gamma_{[P_0,P_1,P_2]}(t)$ be the quadratic Bezier curve with control points $P_0=(0,4)$ and $P_2=(3,-5)$, and all points of $\gamma(t)$ lying on the parabola $y=4-x^2$. Use the fact that the tangent lines to $\gamma(t)$ at the points P_0 and P_2 are determined by the respective line segments through P_1, to find the point P_1.

   a) $(1,3)$ b) $(2,6)$ c) $(\frac{3}{2},4)$ d) $(1,4)$ e) $(\frac{1}{2},\frac{1}{2})$

   Correct Answer: $(\frac{3}{2},4)$

9. Compute the following Vandermonde determinant: $\begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \end{vmatrix}$.

   a) 2 b) 4 c) 0 d) 6 e) 12

   Correct Answer: 0

10. Let $B_i^d(t)=\binom{d}{i}(1-t)^{d-i}t^i$ be the general Bernstein polynomial, and suppose that $\mathbf{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ is the coordinate vector of a polynomial $p(t)$ with respect to the Bernstein basis of P_2. Find the coordinate vector of $p(t)$ with respect to the standard basis $\{1,t,t^2\}$.

    a) $\begin{pmatrix}1\\0\\0\end{pmatrix}$ b) $\begin{pmatrix}0\\-1\\1\end{pmatrix}$ c) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ d) $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ e) $\begin{pmatrix}1\\2\\-1\end{pmatrix}$

    Correct Answer: $\begin{pmatrix}1\\0\\0\end{pmatrix}$

11. Suppose that $\mathbf{v}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ is the coordinate vector of a polynomial $p(t)$ with respect to the standard basis of P_2. Find the coordinate vector of $p(t)$ with respect to the Bernstein basis of P_2. (Hint: solve for the two standard basis vectors that sum up to this vector, since they each have simple Bernstein representations.)

    a) $\begin{pmatrix}0\\0\\1\end{pmatrix}$ b) $\begin{pmatrix}1\\2\\0\end{pmatrix}$ c) $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ d) $\begin{pmatrix}1\\1\\2\end{pmatrix}$ e) $\begin{pmatrix}1\\0\\0\end{pmatrix}$

    Correct Answer: $\begin{pmatrix}1\\1\\2\end{pmatrix}$

12. Let $q(t)$ be a polynomial of degree at most 3 such that $q(0) = 1$, $q'(0) = -1$, $q''(0) = 4$, and $q(2) = 0$. What is $q(1)$? (Don't forget to divide by 2 when entering the second derivative into the divided difference table.)

a) $\frac{1}{8}$ b) $\frac{9}{8}$ c) 1 d) $-\frac{7}{4}$ e) $\frac{7}{4}$

Correct Answer: $\frac{9}{8}$

13. If $p(t) = 2 + 3(t-1) + 5(t-1)(t-2)$ is the Newton form of the interpolating polynomial that matches a data function $g(t)$ for data values $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$, what is $[1, 2, 3]g$?

a) 2 b) 3 c) 5 d) 6 e) 4

Correct Answer: 5

14. Same $p(t)$ and $g(t)$ as in the previous question. Suppose that another interpolation point is added so that $q(t)$ is the new interpolating polynomial in P_3 which satisfies the same information as $p(t)$ and also: $q(4) = -1$. Find $[1, 2, 3, 4]g$:

a) -6 b) -3 c) -4 d) -7 e) -8

Correct Answer: -7

15. Let S be the set $\{(t-1)^2, t, t^2 - 1\}$ of polynomials in P_2. Determine whether the following statements are True or False. The answers are in order i),ii), iii).

i) S is a basis of P_2 ii) S spans P_2 iii) S is linearly independent

a) FFT b) TTT c) FFF d) TFT e) TFF

Correct Answer: TTT

16. Find the derivative: $\frac{d}{dt} B_2^4(t)$.

a) $12t(1-t)(1-2t)$ b) $12t^2(1-2t)$ c) $12t(1-2t)^2$ d) $6t(1-t)^2$ e) $6t(1-t)(1-2t)$

Correct Answer: $12t(1-t)(1-2t)$

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17. Solve for a_1: $(t-3)(t-1)_+^2 = a_1(t-1)_+^2 + a_2(t-1)_+^3$.

a) 1 b) 2 c) -1 d) -2 e) 0

Correct Answer: -2

18. Use a simple observation (about the graph) to find the interpolating polynomial in P_3 that passes through the points $(-1,1)$, $(0,0)$, $(1,1)$ and $(2,4)$. In the standard basis, with $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, what is the coefficient a_3?

a) -1 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 1 e) 0

Correct Answer: 0

19. Same $p(t)$ as in the previous question. What is the coefficient a_2?

a) -1 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 1 e) 0

Correct Answer: 1

20. To show the existence and uniqueness of the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_d t^d$, for a data sequence $t_0, \ldots, t_d$, and data function $g(t)$, with y-values $y_i = g(t_i)$, using the standard basis, we used a linear system $A\mathbf{x} = \mathbf{b}$. The appropriate entries of the column vector $\mathbf{b}$ are:

a) $a_0, \ldots, a_d$ b) $t_0, \ldots, t_d$ c) $y_0, \ldots, y_d$ d) $y_0 - t_0, \ldots, y_d - t_d$ e) $\frac{a_1 - a_0}{t_1 - t_0}, \ldots, \frac{a_d - a_{d-1}}{t_d - t_{d-1}}$

Correct Answer: $y_0, \ldots, y_d$