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MAT 300/500 **Midterm Exam** Spring 2019

Note: For True-False questions, a statement is only True if it must always be True under the given assumptions, otherwise it is False.

1. The control points of a Bezier curve $\gamma(t)$ of degree d are:

   i) values of $\gamma(t)$ for $0 \leq t \leq 1$

   ii) point-coefficients of $\gamma(t)$ with respect to the Bernstein basis

   iii) basis functions for P_d.

   True or False (answers are in the order i),ii),iii))

   a) FTF b) TTT c) TFT d) FFF e) TTF

   Correct Answer: FTF

2. A linear combination of polynomials $p_0(t), \dots, p_n(t)$ is a sum $a_0 p_0(t) + \cdots + a_n p_n(t)$, with real number coefficients a_i. The derivative of a Bernstein polynomial of degree d can be written as a linear combination of:

   i) Bernstein polynomials of degree d

   ii) two Bernstein polynomials of degree $d-1$

   iii) $d+1$ polynomials of the form $(t-t_i)^d$ for some real numbers t_i.

   True or False (answers are in the order i),ii),iii))

   a) FTF b) TTT c) TFT d) FFF e) TTF

   Correct Answer: TTT

3. Let $P_0 = (0,0)$, $P_1 = (2,0)$, $P_2 = (1,1)$, $P_3 = (0,2)$, and $P_4 = (0,0)$, and let $\gamma(t)$ be the Bezier curve with control points P_0, P_1, P_2, P_3, and P_4.

   True or False (answers are in the order i),ii),iii))

   i) $\gamma(t)$ has degree 4 ii) $\gamma(0) = \gamma(1)$ iii) $\gamma'(0) \bullet \gamma'(1) = 0$

   a) TTT b) FFF c) TTF d) FTT e) FTF

   Correct Answer: TTT

4. Same $\gamma(t)$ as in the previous question. Find the Bezier point P_0^2 which is used to compute $\gamma(\frac{1}{2})$. (Sketch the NLI steps.)

   a) $(\frac{5}{8}, \frac{3}{4})$ b) $(\frac{3}{8}, \frac{5}{8})$ c) $(\frac{1}{8}, \frac{7}{8})$ d) $(\frac{5}{4}, \frac{1}{4})$ e) $(\frac{3}{4}, \frac{5}{4})$

   Correct Answer: $(\frac{5}{4}, \frac{1}{4})$

5. Same $\gamma(t)$ as in the previous question. Find $\gamma(\frac{1}{2})$.

   a) $(\frac{5}{8}, \frac{5}{8})$ b) $(\frac{3}{8}, \frac{3}{8})$ c) $(\frac{1}{8}, \frac{1}{8})$ d) $(\frac{7}{8}, \frac{7}{8})$ e) $(\frac{3}{4}, \frac{3}{4})$

   Correct Answer: $(\frac{7}{8}, \frac{7}{8})$

6. Suppose $g(0) = -5$, $g(1) = 3$, and $g(5) = 7$. Find the divided difference: $[0, 1, 5]g$:

   a) $\frac{8}{5}$ b) $-\frac{7}{5}$ c) $\frac{9}{5}$ d) $\frac{7}{5}$ e) $-\frac{9}{5}$

   Correct Answer: $-\frac{7}{5}$

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7. Recall that a function f has a zero of multiplicity $r+1$ at $t=c$ if $f^{(k)}(c)=0$ for $k=0,\dots,r$. We say that f has a zero of *exact* multiplicity $r+1$ if additionally $f^{(r+1)}(c)\neq 0$. Suppose the Bernstein polynomial $B_i^d(t)$ has a zero of exact multiplicity 4 at $t=0$ and a zero of exact multiplicity 3 at $t=1$.

   True or False (answers are in the order i),ii),iii))

   i) $d=4$ ii) $i=3$ iii) $d-i=4$

   a) FFT b) TFF c) FFF d) TTT e) TTF

   Correct Answer: FFF

8. Let $\gamma(t)=\gamma_{[P_0,P_1,P_2]}(t)$ be a quadratic Bezier curve with control points $P_0=(0,4)$ and $P_2=(3,-5)$. Find P_1 if $\gamma(\frac{1}{2})=(\frac{3}{2},\frac{7}{4})$.

   a) $(1,3)$ b) $(2,6)$ c) $(\frac{3}{2},4)$ d) $(1,4)$ e) $(\frac{1}{2},\frac{13}{2})$

   Correct Answer: $(\frac{3}{2},4)$

9. Compute the following Vandermonde determinant: $\begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 6 & 6^2 & 6^3 & 6^4 \end{vmatrix}$.

   a) 2 b) $2^4 3^3$ c) $2^5 3^2$ d) 2^3 e) $2^5 3^3$

   Correct Answer: $2^5 3^2$

10. Let $B_i^d(t)=\binom{d}{i}(1-t)^{d-i}t^i$ be the general Bernstein polynomial, and suppose that $\mathbf{v}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ is the coordinate vector of a polynomial $p(t)$ with respect to the Bernstein basis $\{B_0^2(t),B_1^2(t),B_2^2(t)\}$ of P_2. Find the coordinate vector of $p(t)$ with respect to the standard basis $\{1,t,t^2\}$.

    a) $\begin{pmatrix}1\\0\\0\end{pmatrix}$ b) $\begin{pmatrix}0\\-1\\1\end{pmatrix}$ c) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ d) $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ e) $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

    Correct Answer: $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

11. Suppose that $\mathbf{v}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ is the coordinate vector of a polynomial $p(t)$ with respect to the standard basis of P_2.

    Find the coordinate vector of $p(t)$ with respect to the Bernstein basis of P_2. (Hint: solve for the two standard basis vectors that sum up to this vector, since they each have simple Bernstein representations.)

    a) $\begin{pmatrix}0\\0\\1\end{pmatrix}$ b) $\begin{pmatrix}1\\2\\0\end{pmatrix}$ c) $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ d) $\begin{pmatrix}1\\1\\2\end{pmatrix}$ e) $\begin{pmatrix}1\\0\\0\end{pmatrix}$

    Correct Answer: $\begin{pmatrix}1\\1\\2\end{pmatrix}$

12. Let $q(t)$ be a polynomial of degree at most 3 such that $q(0)=1$, $q'(0)=-1$, $q''(0)=4$, and $q(2)=0$. What is $q(1)$? (Don't forget to divide by 2 when entering the second derivative into the divided difference table.)

    a) $\frac{1}{8}$ b) $\frac{9}{8}$ c) 1 d) $-\frac{7}{4}$ e) $\frac{7}{4}$

    Correct Answer: $\frac{9}{8}$

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13. If $p(t) = 2 + 3(t-1) + 5(t-1)(t-2)$ is the Newton form of the interpolating polynomial that matches a data function $g(t)$ for data values $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$, what is $[1, 2, 3]g$?

a) 2 b) 3 c) 5 d) 6 e) 4

Correct Answer: 5

14. Same $p(t)$ and $g(t)$ as in the previous question. Suppose that another interpolation point is added so that $q(t)$ is the new interpolating polynomial in P_3 which satisfies the same information as $p(t)$ and also: $q(4) = -1$. Find $[1, 2, 3, 4]g$:

a) -6 b) -5 c) -4 d) -7 e) -8

Correct Answer: -7

15. Find a quadratic Bezier curve $\gamma(t)$ which agrees with the graph of $y = x^3$ at the points $(0, 0)$ and $(2, 8)$ and also has the same tangent slopes as this graph at those points. If $\gamma(0) = (0, 0)$ and $\gamma(1) = (2, 8)$ find the control point P_1.

a) $(\frac{4}{3}, 0)$ b) $(\frac{4}{3}, \frac{1}{3})$ c) $(\frac{2}{3}, 0)$ d) $(\frac{3}{4}, 0)$ e) $(\frac{2}{3}, \frac{2}{9})$

Correct Answer: $(\frac{4}{3}, 0)$

16. Let S be the set $\{(t-1)^2, t, t^2+1\}$ of polynomials in P_2. Determine whether the following statements are True or False. The answers are in order i),ii), iii).

i) S is a basis of P_2 ii) S spans P_2 iii) S is linearly independent

a) FFT b) TTT c) FFF d) TFT e) TFF

Correct Answer: FFF

17. Find the derivative: $\frac{d}{dt}B_2^4(t)$.

a) $12t(1-t)(1-2t)$ b) $12t^2(1-2t)$ c) $12t(1-2t)^2$ d) $6t(1-t)^2$ e) $6t(1-t)(1-2t)$

Correct Answer: $12t(1-t)(1-2t)$

18. Solve for a_3: $(t-3)^2(t-1)_+^2 = a_2(t-1)_+^2 + a_3(t-1)_+^3 + a_4(t-1)_+^4$.

a) 4 b) 2 c) -4 d) -2 e) 0

Correct Answer: -4

19. To find an implicit quadratic curve $f(x, y) = L_0(x, y)L_2(x, y) + cL(x, y)^2 = 0$ for a Bezier curve with control points $P_0 = (0, 2)$, $P_1 = (0, 0)$, and $P_2 = (1, 0)$, we could use which of the following implicit forms for $f(x, y)$:

a) $x(x+y) + c(x-2y)^2$ b) $x(x-2y) + c(2x+y-2)^2$ c) $x(y-2) + c(2y-2)^2$ d) $xy + c(2x+y-2)^2$ e) $xy + c(x-y-2)^2$

Correct Answer: $xy + c(2x+y-2)^2$

20. Same curve as in the previous question. Solve for c:

a) 1 b) $\frac{1}{4}$ c) -1 d) $-\frac{1}{4}$ e) $-\frac{1}{8}$

Correct Answer: $-\frac{1}{8}$