Lecture 11

Main Points:

- Osculating polynomial, second proof of existence and uniqueness
- Proof of Newton form for osculating polynomial
- Leibniz' Rule with derivatives

Second proof of existence of osculating polynomial

In the second proof we again appeal to the recursive form and use induction. This time we need to verify the requirement about derivatives. The base case is the same as for interpolation: $d = 0$, with one data value t_0, so $p(t) = [t_0]g = g(t_0)$ is constant. For $d > 0$ the proof breaks into two cases:

a) $t_0 = t_d$ (and thus for all i: $t_0 = t_i = t_d$.)
b) $t_0 < t_d$.

In the first case we simply use the Taylor polynomial from Calculus. This coincides exactly with our definition for the osculating polynomial. This shows existence of the osculating polynomial in the first case. So now we assume $t_0 < t_d$.

For the induction step we assume that $p_0(t)$ and $p_1(t)$ are osculating polynomials with sequences $[t_0, \ldots, t_{d-1}]$ and $[t_1, \ldots, t_d]$ respectively. Then we form the polynomial $p(t)$:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

According to the definition of the osculating polynomial, we now need to verify:

$$p^{(j)}(t_i) = g^{(j)}(t_i), \quad j = 0, \ldots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. In order to check this, we take some derivatives of $p(t)$, to get:

$$p^{(j)}(t) = \frac{t - t_0}{t_d - t_0} p_1^{(j)}(t) + \frac{t_d - t}{t_d - t_0} p_0^{(j)}(t) + j \cdot \frac{p_1^{(j-1)}(t) - p_0^{(j-1)}(t)}{t_d - t_0}.$$

Now to check that $p(t)$ works, assume that we have $t_i = t_{i+1} = \ldots = t_{i+r}$ for some i and r. Case b) from above now breaks into three cases:

i) $t_i = t_0$, ii) $t_i = t_d$ and iii) $t_0 < t_i < t_d$

For case i) we just plug $t_i = t_0$ into the derivative formula $p^{(j)}(t)$ and show that this equals $g^{(j)}(t_0)$, for $j = 0, \ldots, r$. The first two terms give us the correct value, since $p_0^{(j)}(t_0) = g^{(j)}(t_0)$ for $j = 0, \ldots, r$ since the sequence $t_i = t_{i+1} = \ldots = t_{i+r}$ of equal values is part of the sequence for $p_0(t)$. The only slightly tricky part is to show that the last term is zero. This follows from the fact that the sequence $t_1 = t_2 = \ldots = t_r$ has length $r - 1$ and is inside the sequence for $p_1(t)$, so $p_1^{(j-1)}(t_0) = g^{(j-1)}(t_0)$ for $j = 1, \ldots, r$.

The second case is symmetric to the first, and the third case is easier since the the sequences for $p_0(t)$ and $p_1(t)$ both contain the equal values. This completes the existence part of the proof.

Second proof of uniquess of osculating polynomial

For the uniqueness proof we suppose that there are two osculating polynomials $p(t)$ and $q(t)$, and we consider the difference

$$f(t) = p(t) - q(t).$$

Then $f(t)$ is in P_d and we also have:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - q^{(j)}(t_i) = g^{(j)}(t_i) - g^{(j)}(t_i) = 0, \qquad j = 0, \ldots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. By the above fact on multiplicities of zeros for polynomials, this says that whenever there are $r+1$ consecutive equal values $t_i = t_{i+1} = \cdots = t_{i+r}$, then $f(t)$ has a zero of multiplicity $r+1$ at $t = t_i$ and hence also has a factor of the form $(t - t_i)^{r+1}$.

Since there are $d+1$ values in the sequence $t_0, \ldots, t_d$ we can group them into subsequences of equal values. Each such subsequence then corresponds to a factor with exponent equal to the number of terms in the subsequence. The total degree of the product of all such factors is then $d+1$. But $f(t)$ is in P_d and can only have degree at most d or must be zero. So we conclude that $f(t)$ is zero, and hence $p(t) = q(t)$. This completes the uniqueness proof.

Proof of the Newton form for the osculating polynomial

In order to establish the Newton form, we proceed in the same way as for the interpolating polynomial. We assume that $p_0(t)$ is the osculating polynomial with data values $t_0, \ldots, t_{d-1}$ and that $p(t)$ is the osculating polynomial with data values $t_0, \ldots, t_d$. Then we consider the polynomial $f(t) = p(t) - p_0(t)$. This polynomial has the property that

$$f(t_i) = p(t_i) - p_0(t_i) = 0, \qquad i = 0, \ldots, d-1,$$

but many of these values may be repeated. But we also have for the repeated values:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - p_0^{(j)}(t_i) = 0, \qquad j = 0, \ldots, r,$$

whenever $t_i = t_{i+1} = \cdots = t_r$. So, by the fact about multiplicity of zeros, we can write

$$f(t) = p(t) - p_0(t) = C \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where many of the factors may be repeated according to the correct multiplicities implied by the above. We can also identify the constant C by equating coefficients of t^d on both sides to obtain

$$C = [t_0, \ldots, t_d]g.$$

Just as we saw with the interpolating polynomial, we can then write

$$p(t) = q_{d-1}(t) + [t_0, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where $q_{d-1}(t) = p_0(t)$. Repeating this process for $q_{d-1}(t)$ we obtain

$$q_{d-1}(t) = q_{d-2}(t) + [t_0, \ldots, t_{d-1}]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-2})$$

and continuing in the same way we will arrive at the Newton form:

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Leibniz' Rule with repeated values in the divided differences.

Let $f(t) = g(t)h(t)$. Then the divided differences with repeated values also satisfy the Leibniz Rule:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]f = \sum_{r=i}^{i+k} ([t_i, \ldots, t_r]g)([t_r, \ldots, t_{i+k}]h).$$

For degree $d = 2$ it is:

$$[t_0,t_1,t_2]f = [t_0]g[t_0,t_1,t_2]h + [t_0,t_1]g[t_1,t_2]h + [t_0,t_1,t_2]g[t_2]h.$$

We proved this in the case where all the data values t_i were distinct, which used the Newton forms for *interpolating* polynomials. Now we consider the case of repeated data values $t_i = t_{i+1} = \cdots = t_{i+r}$ which is used for Newton forms of *osculating* polynomials.

The base case is still $d = 0$, and since there is only one data value we cannot have any repetition, so it is identical to the previous case. For the rest of the proof, we can proceed exactly as before, using a product of Newton forms for osculating polynomials, and using the uniqueness of the osculating polynomial in P_d.

Examples:

- Define the functions:
$$f(t) = (t-2)_+^2 \cdot \frac{1}{t}, \quad g(t) = (t-2)_+^2, \quad \text{and} \quad h(t) = \frac{1}{t}.$$
We will find the osculating polynomials with the data $[1,1,3,3]$ and data functions $f(t)$, $g(t)$, and $h(t)$. Call these osculating polynomials $p(t)$, $q(t)$, and $r(t)$ respectively. We will verify the Leibniz Rule for $[1,1,3,3]f$. First, we have the following divided difference tables:
$$f(t):\ \begin{array}{ccccc} 1 & 0 & & & \\ & & 0 & & \\ 1 & 0 & & \frac{1}{2} & \\ & & \frac{1}{6} & & \frac{1}{18} \\ 3 & \frac{1}{3} & & \frac{7}{36} & \\ & & \frac{5}{9} & & \\ 3 & \frac{1}{3} & & & \end{array} \qquad g(t):\ \begin{array}{ccccc} 1 & 0 & & & \\ & & 0 & & \\ 1 & 0 & & \frac{1}{4} & \\ & & \frac{1}{2} & & \frac{1}{4} \\ 3 & 1 & & \frac{3}{4} & \\ & & 2 & & \\ 3 & 1 & & & \end{array} \qquad \text{and} \qquad h(t):\ \begin{array}{ccccc} 1 & 1 & & & \\ & & -1 & & \\ 1 & 1 & & \frac{1}{3} & \\ & & -\frac{1}{3} & & -\frac{1}{9} \\ 3 & \frac{1}{3} & & \frac{1}{9} & \\ & & -\frac{1}{9} & & \\ 3 & \frac{1}{3} & & & \end{array}$$
Then we have:
$$[1,1,3,3]f = \frac{1}{18}$$
and
$$\begin{aligned} &[1]g[1,1,3,3]h + [1,1]g[1,3,3]h + [1,1,3]g[3,3]h + [1,1,3,3]g[3]h \\ &= 0 \cdot (-\frac{1}{9}) + 0 \cdot \frac{1}{9} + \frac{1}{4} \cdot (-\frac{1}{9}) + \frac{1}{4} \cdot (\frac{1}{3}) = \frac{1}{18}. \end{aligned}$$
We can also verify that the osculating polynomial $p(t)$ can be found as the appropriate terms taken from the product $q(t)r(t)$. Those polynomials can be obtained with Newton forms. For $p(t)$ and $q(t)$ we use the tables above to get:
$$p(t) = 0 + 0 \cdot (t-1) + \frac{1}{12}(t-1)^2 + \frac{1}{18}(t-1)^2(t-3),$$
$$q(t) = 0 + 0 \cdot (t-1) + \frac{1}{4}(t-1)^2 + \frac{1}{4}(t-1)^2(t-3).$$
For $r(t)$ we do the divided difference table and Newton form in the reverse order, as we did in the proof.
$$\begin{array}{ccccc} 3 & \frac{1}{3} & & & \\ & & -\frac{1}{9} & & \\ 3 & \frac{1}{3} & & \frac{1}{9} & \\ & & -\frac{1}{3} & & -\frac{1}{9} \\ 1 & 1 & & \frac{1}{3} & \\ & & -1 & & \\ 1 & 1 & & & \end{array}$$
So we can write the Newton form:
$$r(t) = \frac{1}{3} - \frac{1}{9}(t-3) + \frac{1}{9}(t-3)^2 - \frac{1}{9}(t-3)^2(t-1).$$

Then we write the terms from $q(t)r(t)$ which consist of products which do not have a zero of multiplicity two at both of the values $t = 1$ and $t = 3$. So, we are avoiding the terms which contain factors $(t-1)^2(t-3)^2$. In the proof we called those terms that we are avoiding $G(t)$ and the remaining terms $F(t)$. Thus we have:

$$F(t) = \frac{1}{3} \cdot \frac{1}{4}(t-1)^2 + \frac{1}{3} \cdot \frac{1}{4}(t-1)^2(t-3) - \frac{1}{9}(t-3)\frac{1}{4}(t-1)^2.$$

At this point, we recall that

$$f(t) = (t-2)_+^2 \cdot \frac{1}{t},$$

and thus

$$f'(t) = 2(t-2)_+^1 \cdot \frac{1}{t} + (t-2)_+^2 \cdot (-\frac{1}{t^2}).$$

We would like to verify that indeed $F(t) = p(t)$. This can be done by checking that $F(t)$ satisfies the osculation conditions for the data $[1, 1, 3, 3]$ with data function $f(t)$. We can check that

$$F(1) = 0 = f(1), \quad \text{and} \quad F(3) = \frac{1}{3} = f(3).$$

Taking a derivative we have:

$$F'(t) = \frac{1}{6}(t-1) + \frac{1}{12}\left[2(t-1)(t-3) + (t-1)^2\right] - \frac{1}{36}\left[(t-1)^2 + 2(t-3)(t-1)\right],$$

and we can also verify that:

$$F'(1) = 0 = f'(1), \quad \text{and} \quad F'(3) = \frac{5}{9} = f'(3).$$

Thus, since $F(t)$ is in P_3 and it satisfies the conditions above, and the osculating polnomial is unique in P_3, we must have $F(t) = p(t)$. Of course, it is also simple to check that the expression for $F(t)$ above can be simplified to obtain:

$$F(t) = \frac{1}{12}(t-1)^2 + \frac{1}{18}(t-1)^2(t-3) = p(t).$$