Lecture 12

Main Points:

- Orders of continuity for splines (piecewise polynomials)
- Standard basis for splines
- Exact orders of continuity for splines
- C^2 cubic splines

Orders of continuity for piecewise polynomials or spline functions

The vector space of piecewise polynomial functions $P_d^k[u_0, \ldots, u_k]$ has many important subspaces of functions which have specific continuity or derivative conditions at the breakpoints $u_1, u_2, \ldots, u_{k-1}$. The simplest of these is the subspace of continuous functions:

$$P_{d,0}^k[u_0, \ldots, u_k] = \{f \in P_d^k[u_0, \ldots, u_k] \ : \ f \text{ is continuous } \}.$$

Note: The subscript zero after the d indicates that the functions must have "zeroth order continuity" which simply means continuity. It is equivalent to specify that at each breakpoint the two polynomials on either side must agree in value, or in other words:

$$p_{i-1}(u_i) = p_i(u_i), \quad i = 1, \ldots, k-1.$$

The next simplest such subspace is the subspace of differentiable functions:

$$P_{d,1}^k[u_0, \ldots, u_k] = \{f \in P_d^k[u_0, \ldots, u_k] \ : \ f \text{ is differentiable } \}.$$

Note: Now the subscript one after the d indicates that the functions must have "first order continuity" which means that they are differentiable and hence also continuous. It is equivalent to specify that at each breakpoint the two polynomials on either side must agree both in value and derivative, or in other words:

$$p_{i-1}(u_i) = p_i(u_i), \quad i = 1, \ldots, k-1,$$

and

$$p'_{i-1}(u_i) = p'_i(u_i), \quad i = 1, \ldots, k-1.$$

The general such subspace is the subspace of functions differentiable to order r:

$$P_{d,r}^k[u_0, \ldots, u_k] = \{f \in P_d^k[u_0, \ldots, u_k] \ : \ f \text{ has } r \text{ continuous derivatives } \}.$$

It is equivalent to specify that at each breakpoint the two polynomials on either side must agree both in value and derivatives up to the r^{th} derivative, or in other words:

$$p_{i-1}^{(j)}(u_i) = p_i^{(j)}(u_i), \quad i = 1, \ldots, k-1, \qquad \text{and} \qquad j = 0, \ldots, r.$$

Examples:

- Determine the appropriate vector spaces to which the function f belongs:

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t + 5, & 0 \le t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \le t \le 4 \end{cases}$$

First, we can say that since f consists of cubic polynomials on the sequence of intervals $[0, 2, 4]$ that f is in the vector space $P_3^2[0, 2, 4]$. Beyond that, we need to know if f is continuous or differentiable to some order. To check continuity we find that

$$p_1(2) = 7 = p_2(2)$$

and thus $f(t)$ is continuous at $t = 2$, so we know also that f is in $P_{3,0}^2[0, 2, 4]$. Next, we find the following derivatives:

$$p_1'(t) = 3t^2 - 4t + 1,\ p_1'(2) = 5, \quad p_2'(t) = 6t^2 - 16t + 13,\ p_2'(2) = 5$$
$$p_1''(t) = 6t - 4,\ p_1''(2) = 8, \quad p_2''(t) = 12t - 16,\ p_2''(2) = 8$$
$$p_1'''(t) = 6,\ p_1'''(2) = 6, \quad p_2'''(t) = 12,\ p_2'''(2) = 12.$$

We can see that the derivatives match at $t = 2$ up to order 2, so the function f has two continuous derivatives and is in the vector space $P_{3,2}^2[0, 2, 4]$. Note: this last vector space is the most specific, since it requires two continuous derivatives. This is a subspace of $P_{3,1}^2[0, 2, 4]$, which also contains f. In fact, all of these vector space can be put into the sequence of subspaces:

$$P_{3,2}^2[0, 2, 4] \subset P_{3,1}^2[0, 2, 4] \subset P_{3,0}^2[0, 2, 4] \subset P_3^2[0, 2, 4].$$

Standard basis and dimension of $P_{d,r}^k[u_0, \ldots, u_k]$:

The standard basis of $P_{d,r}^k[u_0, \ldots, u_k]$ is constructed from the standard basis of P_d together with shifted power functions of degree d down to degree $r + 1$:

$$\{1, t, t^2, \ldots, t^d, (t - u_1)_+^d, \ldots, (t - u_1)_+^{r+1}, \ldots, (t - u_{k-1})_+^d, \ldots, (t - u_{k-1})_+^{r+1}\}.$$

The dimension of $P_{d,r}^k[u_0, \ldots, u_k]$ is thus:

$$\dim(P_{d,r}^k[u_0, \ldots, u_k]) = d + 1 + (d - r)(k - 1).$$

Examples:

- Let $V = P_{3,2}^4[0, 1, 2, 3, 4]$. Then V has basis $\{1, t, t^2, t^3, (t - 1)_+^3, (t - 2)_+^3, (t - 3)_+^3\}$ and dimension 7.
- Let $V = P_{2,0}^4[0, 1, 2, 3, 4]$. Then V has basis $\{1, t, t^2, (t - 1)_+^2, (t - 1)_+^1, (t - 2)_+^2, (t - 2)_+^1, (t - 3)_+^2, (t - 3)_+^1\}$ and dimension 9.
- Let $V = P_{4,1}^3[0, 1, 2, 3]$. Then V has basis $\{1, t, t^2, t^3, t^4, (t - 1)_+^4, (t - 1)_+^3, (t - 1)_+^2, (t - 2)_+^4, (t - 2)_+^3, (t - 2)_+^2\}$ and dimension 11.
- The function f from a previous example

  $$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t + 5, & 0 \leq t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \leq t \leq 4 \end{cases}$$

  was shown to be in the vector space $P_{3,2}^2[0, 2, 4]$. We can express f as a linear combination of the basis:

  $$\{1, t, t^2, t^3, (t - 2)_+^3\},$$

  we need to find coefficients so that

  $$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 (t - 2)_+^3.$$

  Clearly, the first 4 coefficients determine the polynomial $p_1(t)$ on the first interval $[0, 2)$. This means we have:

  $$f(t) = 5 + t - 2t^2 + t^3 + a_4 (t - 2)_+^3.$$

To obtain the correct values for f on the second subinterval, we need to have

$$a_4(t-2)^3_+ = p_2(t) - p_1(t) = t^3 - 6t^2 + 12t - 8 = (t-2)^3,$$

and so we see that $a_4 = 1$ and we have:

$$f(t) = 5 + t - 2t^2 + t^3 + (t-2)^3_+.$$

Exact order of continuity of shifted power functions

We say a function $f(t)$ has order of continuity r at $t = c$ if f is continuous at $t = c$ and each of the derivative functions $f', f'', \ldots, f^{(r)}$ are continuous at $t = c$. If, in addition, the function $f^{(r+1)}$ is *not* continuous at $t = c$, then we say that f has *exact order of continuity* r at $t = c$. If f and all of its derivatives are continuous at $t = c$ then we say f has infinite order of continuity, or simply f is continuous to all orders at $t = c$.

The shifted power function $(t-c)^k_+$ is continuous to all orders at all points not equal to c, and can be seen to have exact order of continuity $k-1$ at $t = c$.

If we let $f(t) = (t-c)^k_+$ then the derivatives of f are:

$$f'(t) = k(t-c)^{k-1}_+, f''(t) = k(k-1)(t-c)^{k-2}_+, \ldots, f^{(k-1)}(t) = k!(t-c)^1_+.$$

Note: The function $f(t) = (t-c)^1_+$ is not differentiable at $t = c$, although it is continuous there. The function $f(t) = (t-c)^0_+$ is neither continuous nor differentiable at $t = c$.

Note: The standard basis of $P^k_{d,r}[u_0, \ldots, u_k]$ can be obtained from the standard basis of $P^k_d[u_0, \ldots, u_k]$ by simply removing the functions which have the incorrect orders of continuity. In particular, the functions $(t-c)^r_+$ have order of continuity $r-1$ which is one lower than required. Hence all of those functions and lower powers do not belong. This is not a general rule, but rather, it is a special property of this particular collection of bases. Other collections of bases can be found at the opposite extreme, for instance the first basis of ordered k-tuples that we gave for P^k_d corresponds only to discontinuous functions. Thus it is impossible to remove any functions from this basis and end up with a basis of $P^k_{d,r}[u_0, \ldots, u_k]$.

Examples:

- The function $f(t) = (t-2)^2_+$ has exact order of continuity 1 at $t = 2$. This means that the first derivative $f'(t) = 2(t-2)^1_+$ exists and is continuous at $t = 2$ but that the second derivative fails to exist at $t = 2$. Note that it is tempting to say that since the function can be differentiated almost everywhere (except at $t = 2$) and can be given by the formula $2(t-2)^0_+$ away from $t = 2$, that this is the second derivative of f. But the function $2(t-2)^0_+$ is defined and equal to 2 at $t = 2$, but $f'(t)$ does not have a value there, so the two functions do not agree, and f' fails to be differentiable.
- The standard basis of $P^3_2[0, 2, 4, 6]$ is

  $$\{1, t, t^2, (t-2)^2_+, (t-2)^1_+, (t-2)^0_+, (t-4)^2_+, (t-4)^1_+, (t-4)^0_+\}.$$

  In order to obtain a basis for $P^3_{2,0}[0, 2, 4, 6]$ we can simply throw out the discontinuous functions from the previous basis to obtain:

  $$\{1, t, t^2, (t-2)^2_+, (t-2)^1_+, (t-4)^2_+, (t-4)^1_+\}.$$

  In order to obtain a basis for $P^3_{2,1}[0, 2, 4, 6]$ we can simply throw out the nondifferentiable functions from the previous basis to obtain:

  $$\{1, t, t^2, (t-2)^2_+, (t-4)^2_+\}.$$

- A different basis of $P_2^3[0,2,4,6]$ is the one that corresponds to the set of ordered triples:

$$\{(1,0,0),(t,0,0),(t^2,0,0),(0,1,0),(0,t,0),(0,t^2,0),(0,0,1),(0,0,t),(0,0,t^2)\}$$

  Note that the functions corresponding to these triples, defined on the sequence of intervals $[0,2,4,6]$, are *all discontinuous* and hence no subset of this basis will be a basis of the continuous or differentiable subspaces above.

Proof of standard basis for $V = P_{d,r}^k[u_0,\ldots,u_k]$.

We will verify that the standard basis spans V and is linearly independent. For the spanning property we need to show that any function in V can be written as a linear combination of the basis functions. Let $f \in V$. Then f is a piecewise polynomial which corresponds to a k-tuple of polynomial functions $(p_1,p_2,\ldots,p_k)$ with the property:

$$p_{i-1}^{(j)}(u_i) = p_i^{(j)}(u_i), \quad i = 1,\ldots,k-1, \qquad \text{and} \qquad j = 0,\ldots,r.$$

Recall that we can write such a k-tuple as:

$$\begin{aligned}(p_1,p_2,\ldots,p_k) \quad = \quad & (p_1,p_1,\ldots,p_1) \\ & + (0,p_2-p_1,\ldots,p_2-p_1) \\ & + (0,0,p_3-p_2\ldots,p_3-p_2) \\ & \vdots \\ & + (0,0,0\ldots,0,p_k-p_{k-1}).\end{aligned}$$

We will show that each line in the above sum can be generated by a particular set of k-tuples, which in turn corresponds to a particular set of piecewise polynomial functions.

The first k-tuple $(p_1,\ldots,p_1)$ is easily generated as a sum of k-tuples of the form:

$$(1,1,\ldots,1),(t,t,\ldots,t),\ldots,(t^d,t^d,\ldots,t^d).$$

As functions, these correspond to the polynomials $1,t,t^2,\ldots,t^d$.

Next, we note that any difference $q_i = p_i - p_{i-1}$ above is a polynomial which satisfies:

$$q_i^{(j)}(u_i) = 0, \ j = 0,\ldots,r.$$

But this means that the function $q_i(t)$ has a zero of multiplicity r at $t = u_i$, which means that $(t-u_i)^r$ is a factor of q_i. Thus we can write:

$$q_i(t) = a_1(t-u_i)^{r+1} + a_2(t-u_i)^{r+2} + \cdots + a_{d-r}(t-u_i)^d,$$

which shows that $q_i(t)$ is a linear combination of a subset of a shifted basis of P_d. This is a linearly independent set (since it is a subset of a basis) and gives us the recipe for generating the ordered k-tuples of the form $(0,0,\ldots,0,q_i(t),q_i(t),\ldots,q_i(t))$. In fact, all such elements are obtained as linear combinations of:

$$(0,0,\ldots,0,(t-u_i)^{r+1},\ldots,(t-u_i)^{r+1}),(0,0,\ldots,0,(t-u_i)^{r+2},\ldots,(t-u_i)^{r+2}),\ldots,(0,0,\ldots,0,(t-u_i)^d,\ldots,(t-u_i)^d).$$

But these k-tuples correspond exactly to the shifted power functions

$$(t-u_i)_+^{r+1},(t-u_i)_+^{r+2},\ldots,(t-u_i)_+^d.$$

Thus any function in V corresponding to a k-tuple $(p_1,\ldots,p_k)$ can be represented as a sum of the functions in the suggested basis. So we know that this set spans V.

To check linear independence, we suppose that we have a linear combination of the basis elements equal to the zero function:

$$\begin{aligned} a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d \quad &+ \quad b_{1,r+1}(t-u_1)_+^{r+1} + \cdots + b_{1,d}(t-u_1)_+^d \\ &+ \quad b_{2,r+1}(t-u_2)_+^{r+1} + \cdots + b_{2,d}(t-u_2)_+^d \\ &\quad\vdots \\ &+ \quad b_{k-1,r+1}(t-u_{k-1})_+^{r+1} + \cdots + b_{k-1,d}(t-u_{k-1})_+^d = 0. \end{aligned}$$

Now it is straight-forward to check that all the coefficients must be zero, by simply focusing on each subinterval from left to right. Since any shifted power function $(t-c)_+^j$ is zero for $t < c$, we can see that on the first subinterval $[u_0, u_1)$ the sum consists only of the polynomial $a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$. Since the equation above must true on each subinterval, we can restrict the t-values to the subinterval $[u_0, u_1)$ to obtain:

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d = 0, \quad \text{in } P_d.$$

But this is a sum of the standard basis, which is certainly independent in P_d, so all the coefficients a_i must be equal to zero. The above sum now reduces to:

$$\begin{aligned} b_{1,r+1}(t-u_1)_+^{r+1} + \cdots + b_{1,d}(t-u_1)_+^d \quad &+ \quad b_{2,r+1}(t-u_2)_+^{r+1} + \cdots + b_{2,d}(t-u_2)_+^d \\ &\quad\vdots \\ &+ \quad b_{k-1,r+1}(t-u_{k-1})_+^{r+1} + \cdots + b_{k-1,d}(t-u_{k-1})_+^d = 0. \end{aligned}$$

Next, we consider the subinterval $[u_1, u_2)$. The only functions remaining in the above sum which are nonzero on this interval are those of the type $(t-u_1)_+^j$. Further, on the interval $[u_1, u_2)$ these functions are simply polynomials $(t-u_1)^j$. So we obtain:

$$b_{1,r+1}(t-u_1)^{r+1} + \cdots + b_{1,d}(t-u_1)^d = 0, \quad \text{in } P_d.$$

But these polynomials are a subset of a shifted basis of P_d, so are certainly linearly independent in P_d, hence all of these coefficients $b_{i,r+1}$ must also be zero.

Proceeding this way, on each subinterval $[u_i, u_{i+1})$ we conclude that all of the coefficients must be zero and the suggested basis is indeed linearly independent, and hence a basis of V.