Lecture 13

Main Points:

- Review for Midterm Exam
- Continuity Conditions for Splines

Points to be able to work out for midterm exam:

- Determine if a set of polynomials is linearly independent or not using a determinant of the matrix of coordinate vectors
- Determine if a set of polynomials is a top-down basis or not
- Write the coordinate vector of a polynomial expressed as a linear combination of basis polynomials.
- Find the change of basis matrix to go from one polynomial basis to another
- Find the coefficients of an interpolating polynomial in the standard basis using a linear system
- Compute a Vandermonde or Confluent Vandermonde determinant
- Apply the properties of Bernstein polynomials
- Convert a Bezier curve from standard to BB-form using change of basis for coordinate polynomials
- Write bases for P_d^k with ordered k-tuples
- Write bases for $P_d^k[u_0, \ldots, u_k]$ with shifted power functions
- Convert between $P_d^k[u_0, \ldots, u_k]$ and P_d^k using the correspondence between shifted power functions and ordered k-tuples
- Write a piecewise polynomial function in terms of a basis
- Find an interpolating polynomial using Lagrange polynomials
- Find an interpolating polynomial using Newton form
- Find an osculating polynomial using Newton form
- Compute a divided difference given information about an interpolating or osculating polynomial $p(t)$ by using the definition of divided difference
- Compute a divided difference using a table and the recursive formula
- Find an interpolating or osculating polynomial given some data by using the existence and uniqueness theorem
- Compute a divided difference with the Leibniz Rule and verify directly
- Switch the sequence of values for divided differences in the interpolating or osculating cases without changing the resulting polynomial
- Verify the induction step for interpolating or osculating polynomial of degree d based on polynomials of degree $d-1$
- Determine the exact order of continuity of a shifted power function
- Determine the exact order of continuity of a spline function in piecewise polynomial form
- Write a basis for a vector of continuous splines by removing basis functions from a previous basis

Examples of continuity conditions for splines:

- Let $V = P^4_{3,1}[0,1,2,3,4]$. Then V has basis $\{1, t, t^2, t^3, (t-1)^3_+, (t-1)^2_+, (t-2)^3_+, (t-2)^2_+, (t-3)^3_+ (t-3)^2_+\}$ and dimension 10.

- For the previous V, determine if the function

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t, & 0 \leq t < 2 \\ p_2(t) = t^3 + 3t^2 - 14t + 12, & 2 \leq t \leq 4 \end{cases}$$

  is in V, or not. We check and find that $p_1(2) = p_2(2) = 4$ and also that $p_1'(2) = p_2'(2) = 10$, but that $p_1''(2) = 12 \neq p_2''(2) = 18$. This confirms that f is indeed in V.

- Write the previous function f in terms of the standard basis of V. We need to find:

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4(t-1)^3_+ + a_5(t-1)^2_+ + a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Clearly, the first four coefficients determine $p_1(t)$, so we have:

$$f(t) = -2t + t^3 + a_4(t-1)^3_+ + a_5(t-1)^2_+ + a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Next, we note that there is no need to change this function on the second subinterval $[1,2)$, so we do not need the functions $a_4(t-1)^3_+$ or $a_5(t-1)^2_+$, and so

$$f(t) = -2t + t^3 + +a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Now we need to use the next functions to obtain $p_2(t) - p_1(t)$ on the subinterval $[2,3)$. So we compute:

$$p_2(t) - p_1(t) = (t^3 + 3t^2 - 14t + 12) - (t^3 - 2t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t-2)^2.$$

  So when we restrict the previous version of f to the interval $[2,3)$ we obtain

$$p_2(t) = p_1(t) + a_6(t-2)^3 + a_7(t-2)^2$$

  or

$$p_2(t) - p_1(t) = a_6(t-2)^3 + a_7(t-2)^2$$

  or

$$3(t-2)^2 = a_6(t-2)^3 + a_7(t-2)^2.$$

  But this has the obvious solution: $a_6 = 0$ and $a_7 = 3$. Next, we notice that the function f does not change from the interval $[2,3)$ to the interval $[3,4]$, so we do not need the functions $a_8(t-3)^3_+$ or $a_9(t-3)^2_+$ and we can take $a_8 = a_9 = 0$. Finally, we have:

$$f(t) = -2t + t^3 + 3(t-2)^2_+.$$

- The function $f(t) = (t-1)^3_+$ has exact order of continuity 2 at $t = 1$. This means that the first derivative $f'(t) = 3(t-2)^2_+$ and second derivative $f''(t) = 6(t-2)^1_+$ both exist and are continuous at $t = 1$ but that the third derivative fails to exist at $t = 1$. The derivative of $f''(t) = 6(t-2)^1_+$ exists almost everywhere and is equal to $6(t-2)^0_+$ except at $t = 1$, but $f''(t)$ fails to have a derivative at $t = 1$, so f is continuous to exact order 2, since f'' fails to be differentiable at $t = 1$.

- The standard basis of $P^3_1[0,2,4,6]$ is

$$\{1, t, (t-2)^1_+, (t-2)^0_+, (t-4)^1_+, (t-4)^0_+\}.$$

  In order to obtain a basis for $P^3_{1,0}[0,2,4,6]$, the *continuous* piecewise linear functions, we can simply throw out the discontinuous functions from the previous basis to obtain:

$$\{1, t, (t-2)^1_+, (t-4)^1_+\}.$$

- A different basis of $P^3_1[0,2,4,6]$ is the one that corresponds to the set of ordered triples:

$$\{(1,0,0), (t,0,0), (0,1,0), (0,t,0), (0,0,1), (0,0,t)\}$$

  Note that the functions corresponding to these triples, defined on the sequence of intervals $[0,2,4,6]$, are *all discontinuous* and hence no subset of this basis will be a basis of the continuous subspace above.