# Lecture 14

Main Points:

- Review of bases for splines with continuity
- Affine functions
- Polar forms for polynomials
- Polar forms for Bezier curves

### Spline vector spaces with continuity

Recall that the vector space $P^k_{d,r}[u_0, \dots, u_k]$ of piecewise polynomial functions, also called spline functions, has standard basis given by the union of the sets: i) standard basis of $P_d$, ii) the shifted power functions $(t - u_i)^j_+$, $j = r+1, \dots, d$, $i = 1, \dots, k-1$. For example, $P^4_{3,1}[0, 1, 2, 3, 4]$ has basis:

$$\{1, t, t^2, t^3, (t-1)^2_+, (t-1)^3_+, (t-2)^2_+, (t-2)^3_+, (t-3)^2_+, (t-3)^3_+\}.$$

### Affine functions

An affine function $f(x)$ can be defined as:

$$f(x) = \alpha x + \beta$$

where $\alpha$ and $\beta$ are constants.

An important property of affine functions is that they respect affine sums. In particular:

$$f((1-t)a + tb) = (1-t)f(a) + tf(b).$$

### Polar forms for polynomials

A polar form $F[u_1, \dots, u_d]$ for a polynomial $f(t)$ in $P_d$ is a multivariable multi-valued function satisfying the following properties:

- (symmetry) $F[u_{\sigma(1)}, \dots, u_{\sigma(d)}] = F[u_1, \dots, u_d]$ for any permutation $\sigma$ of the set $\{1, 2, \dots, d\}$.
- (substitution) $F[t, \dots, t] = f(t)$
- (affine) $F[u_1, \dots, x, \dots, u_d]$ is an affine function of $x$, where $x$ replaces the variable $u_i$, for any $i$, and the other $u_j$ are treated as constant. The coefficients in the affine function then depend on the the $u_j$ for $j \neq i$.

### Examples:

- The polar form of $f(t) = 1$ in $P_d$ is $F[u_1, \dots, u_d] = 1$.
- The polar form of $f(t) = t$ in $P_d$ is

$$F[u_1, \dots, u_d] = \frac{u_1 + u_2 + \cdots + u_d}{d}.$$

- The polar form of $f(t) = t^d$ in $P_d$ is

$$F[u_1, \dots, u_d] = u_1 u_2 \cdots u_d.$$

- The polar form of $f(t) = t^2$ in $P_3$ is

$$F[u_1, u_2, u_3] = \frac{u_1u_2 + u_1u_3 + u_2u_3}{3}.$$

- The polar form of $f(t) = t^3$ in $P_4$ is

$$F[u_1, u_2, u_3, u_4] = \frac{u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4}{4}.$$

**Polar forms for the standard basis**

We can find polar forms for any polynomial in the standard basis of $P_d$ as indicated in the above examples. For $f(t) = t^j$ in $P_d$, we simply form the sum of all products of exactly $j$ variables from the list $u_1, \ldots, u_d$, and then divide by $\binom{d}{j}$. We can write this as:

$$F[u_1, \ldots, u_d] = \frac{u_1u_2 \cdots u_j + \cdots + u_{d-j+1}u_{d-j} \cdots u_d}{\binom{d}{j}}.$$

**Polar forms for Bernstein basis**

For $f(t) = B_i^d(t) = \binom{d}{i}(1-t)^{d-i}t^i$, we can form the polar form $F[u_1, \ldots, u_d]$ which consists of the sum of all terms which are products of $d-i$ factors of the type $(1-u_j)$ and $i$ factors of the type $u_j$. We can write this as:

$$\begin{aligned} F[u_1, \ldots, u_d] &= (1-u_1)(1-u_2)\cdots(1-u_{d-i})u_{d-i+1}\cdots u_d \\ &\vdots \\ &+ (1-u_{i+1})(1-u_{i+2}\cdots(1-u_d)u_1u_2\cdots u_i. \end{aligned}$$

**Examples:**

- The polar form of $f(t) = B_1^2(t) = 2(1-t)t$ in $P_2$ is

$$F[u_1, u_2] = (1-u_1)u_2 + (1-u_2)u_1.$$

- The polar form of $f(t) = B_1^3(t) = 3(1-t)^2t$ in $P_3$ is

$$F[u_1, u_2, u_3] = (1-u_1)(1-u_2)u_3 + (1-u_1)(1-u_3)u_2(1-u_2)(1-u_3)u_1.$$

- The polar form of $f(t) = B_2^4(t) = 6(1-t)^2t^2$ in $P_4$ is

$$\begin{aligned} F[u_1, u_2, u_3, u_4] &= (1-u_1)(1-u_2)u_3u_4 + (1-u_1)(1-u_3)u_2u_4 + (1-u_1)(1-u_4)u_2u_3 \\ &+(1-u_2)(1-u_3)u_1u_4 + (1-u_2)(1-u_4)u_1u_3 + (1-u_3)(1-u_4)u_1u_2. \end{aligned}$$

**Polar forms for any polynomial**

We define a polar form for a general polynomial to be the sum of polar forms with coefficients with respect to a basis. Specifically, if

$$f(t) = a_0 + a_1t + a_2t^2 + \cdots + a_dt^d,$$

then a polar form is:

$$F[u_1, \ldots, u_d] = a_0F_0 + a_1F_1 + \cdots + a_dF_d,$$

where each $F_i$ is a polar form for $t^i$ in $d$ variables $u_1, \ldots, u_d$.

**Examples:**

- A polar form for $f(t) = 2 - 3t + 4t^2$ is:

$$F[u_1, u_2] = 2 - 3\left(\frac{u_1 + u_2}{2}\right) + 4u_1u_2.$$

**Existence and Uniqueness Theorem for Polar Forms of polynomials**

For any polynomial $f(t)$ in $P_d$, there is exactly one multi-variable function $F[u_1, \ldots, u_d]$, called the polar form of $f$, which satisfies the three defining properties.