Lecture 15

Main Points:

- Polar forms for Bezier curves

Polar forms for parametric curves

For $\gamma(t) = (p_1(t), \ldots, p_n(t))$ a parametric polynomial curve, we define a polar form for $\gamma(t)$ to be given by the polar forms for each component polynomial. It is convenient to put the curve into point coefficient form to reduce the repetition of functions on the coordinates.

Examples:

- Let $\gamma(t)$ be defined in parametric form, and point-coefficient form with respect to the standard basis:

  $$\gamma(t) = (1 - t^2, 3 + t + 2t^2) = (1,3) + (0,1)t + (-1,2)t^2.$$

  Then the polar form of $\gamma(t)$ is:

  $$F[u_1, u_2] = (1,3) + (0,1)\frac{u_1 + u_2}{2} + (-1,2)u_1 u_2$$

- Let $\gamma(t)$ be defined in BB-form, which is point-coefficient form with respect to the Bernstein basis:

  $$\gamma(t) = (1-t)^2(2,-1) + 2(1-t)t(3,4) + t^2(1,3).$$

  Then the polar form of $\gamma(t)$ is:

  $$F[u_1, u_2] = ((1-u_1)(1-u_2))\,(2,-1) + ((1-u_1)u_2 + (1-u_2)u_1)\,(3,4) + u_1 u_2 (1,3).$$

Control Point Property for a parametric polynomial curve

If $\gamma(t)$ is a parametric polynomial curve with polar form $F[u_1, \ldots, u_d]$, then the control points of $\gamma(t)$ are given by:

$$P_i = F[0, 0, \ldots, 0, 1, 1, \ldots, 1],$$

where the number of 1's is equal to the subscript i, for $i = 0, \ldots, d$.

Reparametrization Property for a parametric polynomial curve

If $\gamma(t)$ is a parametric polynomial curve with polar form $F[u_1, \ldots, u_d]$, and $\alpha(t) = \gamma((1-t)a + tb)$ is a reparametrization of $\gamma(t)$, then the control points of $\alpha(t)$ are given by:

$$P_i = F[a, a, \ldots, a, b, b, \ldots, b],$$

where the number of b's is equal to the subscript i, for $i = 0, \ldots, d$.

Examples:

- Polar Form of a quadratic Bezier curve $\gamma_{[P_0,P_1,P_2]}(t)$:

$$F[u_1, u_2] = (1 - u_1)(1 - u_2)P_0 + [(1 - u_1)u_2 + (1 - u_2)u_1]P_1 + u_1u_2P_2.$$

- Control point property: $F[0, 0] = P_0$, $F[0, 1] = P_1$, $F[1, 1] = P_2$.
- Reparamatrization property: $F[a, a] = Q_0$, $F[a, b] = Q_1$, $F[b, b] = Q_2$, where Q_0, Q_1, and Q_2 are the control points of $\alpha(t) = \gamma((1 - t)a + tb)$.
- Polar Form of a cubic Bezier curve $\gamma_{[P_0,P_1,P_2,P_3]}(t)$:

$$F[u_1, u_2, u_3] = (1 - u_1)(1 - u_2)(1 - u_3)P_0$$
$$+[(1 - u_1)(1 - u_2)u_3 + (1 - u_1)(1 - u_3)u_2 + (1 - u_2)(1 - u_3)u_1]P_1$$
$$+[(1 - u_1)u_2u_3 + (1 - u_2)u_1u_3 + (1 - u_3)u_1u_2]P_2 + u_1u_2u_3P_3.$$

- Control point property: $F[0, 0, 0] = P_0$, $F[0, 0, 1] = P_1$, $F[0, 1, 1] = P_2$, $F[1, 1, 1] = P_3$.
- Reparamatrization property: $F[a, a, a] = Q_0$, $F[a, a, b] = Q_1$, $F[a, b, b] = Q_2$, $F[b, b, b] = Q_3$, where Q_0, Q_1, Q_2 and Q_3 are the control points of $\alpha(t) = \gamma((1 - t)a + tb)$.

Main Theorem on Polar Forms for parametric curves (Existence and Uniqueness)

Every parametric polynomial curve $\gamma(t)$ has a unique polar form $F[u_1, \dots, u_d]$ which satisfies the defining properties above, and further satisfies the control point property and reparametrization property.

Polar forms can be evaluated by Nested Linear Interpolation

In order to prove the uniqueness property, as well as the control point and reparametrization properties, of polar forms, we first need to show that they can be evaluated by Nested Linear Interpolation.

First recall the Nested Linear Interpolation diagram of Bezier points for a degree 2 Bezier curve $\gamma(t)$:

$$\begin{array}{ccc} P_0^0 & & \\ & P_0^1 & \\ P_1^0 & & P_0^2 = \gamma(t) \\ & P_1^1 & \\ P_2^0 & & \end{array}$$

Any small triangle of points in this diagram such as:

$$\begin{array}{cc} P_0^0 & \\ & P_0^1 \\ P_1^0 & \end{array}$$

is meant to imply that

$$(1 - t)P_0^0 + tP_1^0 = P_0^1,$$

for example. Now we will construct a similar table of values of $F[u_1, u_2]$, assuming F is a polar form for $\gamma(t)$:

$$\begin{array}{ccc} F[0,0] & & \\ & F[u_1,0] & \\ F[0,1] & & F[u_1,u_2] \\ & F[u_1,1] & \\ F[1,1] & & \end{array}$$

In this diagram, each small triangle of points represents a nested linear interpolation, but not with parameter t. In this case, the values in the second column are obtained with parameter u_1, and the value in the third column is obtained with parameter u_2. Specifically:

$$(1 - u_1)F[0, 0] + u_1F[0, 1] = F[u_1, 0], \qquad (1 - u_1)F[0, 1] + u_1F[1, 1] = F[u_1, 1],$$

$$\text{and} \qquad (1-u_2)F[u_1,0]+u_2F[u_1,1]=F[u_1,u_2].$$

To see that these relationships are valid, we use the properties of the polar form. For instance, since a polar form is affine in each coordinate, we know that it respects affine sums in either coordinate in the following way:

$$F[(1-s)a+sb,c]=(1-s)F[a,c]+sF[b,c], \qquad \text{and} \qquad F[c,(1-s)a+sb]=(1-s)F[c,a]+sF[c,b].$$

Using the first of these with $s=u_1$, $a=0$, $b=1$, and $c=1$, we get:

$$(1-u_1)F[0,1]+u_1F[1,1]=F[u_1,1].$$

The other two can be obtained similarly, also using the symmetry property: $F[u_1,u_2]=F[u_2.u_1]$. For instance, with $s=u_1$, $a=0$, $b=1$, and $c=0$, in the second equation, we get:

$$(1-u_1)F[0,0]+u_1F[0,1]=F[0,u_1]=F[u_1,0].$$

Finally, with $s=u_2$, $a=0$, $b=1$, and $c=u_1$, in the second equation, we get:

$$(1-u_2)F[u_1,0]+u_2F[u_1,1]=F[u_1,u_2].$$

More generally, it follows for any polar form $F[u_1,\dots,u_d]$, that F can be evaluated by nested linear interpolation in stages with parameters $u_1,u_2\dots u_d$, from the starting values

$$F[0,0,\dots,0],F[0,0,\dots,0,1],F[0,0,\dots,0,1,1],\dots,F[0,1,\dots,1,1],F[1,1,\dots,1,1].$$

By combining the affine and symmetry properties, as in the above example for $d=2$, we arrive at the same conclusion.

Equivalence of Nested Linear Interpolation (NLI) and BB-form for Bezier curves

For degree $d=1$, the NLI form for a Bezier curve is:

$$\gamma(t)=(1-t)P_0+tP_1.$$

This is identical to the BB-form, since the Bernstein polynomials for degree $d=1$ are simply:

$$B_0^1(t)=1-t, \qquad \text{and} \qquad B_1^1(t0=t.$$

For degree $d=2$, the NLI form for a Bezier curve is:

$$\begin{aligned}
\gamma(t) &= (1-t)\left[(1-t)P_0+tP_1\right]+t\left[(1-t)P_1+tP_2\right]\\
&= (1-t)^2P_0+(1-t)tP_1+t(1-t)P_1+t^2P_2\\
&= (1-t)^2P_0+2(1-t)tP_1+t^2P_2\\
&= B_0^2(t)P_0+B_1^2(t)P_1+B_2^2(t)P_2.
\end{aligned}$$

which is also identical to the BB-form.

In general, we can write the NLI form for a curve $\gamma(t)$ with control points $P_0,\dots,P_d$ recursively as:

$$\gamma(t)=\gamma_{[P_0,\dots,P_d]}(t)=(1-t)\gamma_{[P_0,\dots,P_{d-1}]}(t)+t\gamma_{[P_1,\dots,P_d]}(t).$$

This form has the same content as the Bezier point table, where the parameter t is implied. In fact, the final triangle in that table:

$$\begin{array}{ccc}
P_0^{d-1} & & \\
 & & P_0^d \\
P_1^{d-1} & &
\end{array}$$

is equivalent to the statement:

$$\gamma(t)=P_0^d=(1-t)P_0^{d-1}+tP_1^{d-1}=(1-t)\gamma_{[P_0,\dots,P_{d-1}]}(t)+t\gamma_{[P_1,\dots,P_d]}(t).$$

We also have the BB-form with the same control points as:

$$\gamma(t) = B_0^d(t)P_0 + B_1^d(t)P_1 + \cdots + B_d^d(t)P_d = \sum_{i=0}^{d} B_i^d(t)P_i.$$

To see that these are the same, we use induction the recursive formula for the Bernstein polynomials. For the induction hypothesis we assume that these are equivalent for degree k with $0 \le k \le d-1$:

$$\gamma_{[P_0,\ldots,P_k]}(t) = \sum_{i=0}^{d} B_i^k(t)P_i.$$

Then we have:

$$\begin{aligned}
\gamma_{[P_0,\ldots,P_d]}(t) &= (1-t)\gamma_{[P_0,\ldots,P_{d-1}]}(t) + t\gamma_{[P_1,\ldots,P_d]}(t)\\
&= (1-t)\sum_{i=0}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=0}^{d-1} B_i^{d-1}(t)P_{i+1}\\
&= (1-t)^dP_0 + \sum_{i=1}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=0}^{d-2} B_i^{d-1}(t)P_{i+1} + t^dP_d\\
&= (1-t)^dP_0 + (1-t)\sum_{i=1}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=1}^{d-1} B_{i-1}^{d-1}(t)P_i + t^dP_d\\
&= (1-t)^dP_0 + \sum_{i=1}^{d-1} \left[(1-t)B_i^{d-1}(t) + tB_{i-1}^{d-1}(t)\right]P_i + t^dP_d\\
&= (1-t)^dP_0 + \sum_{i=1}^{d-1} \left[(1-t)B_i^{d-1}(t) + tB_{i-1}^{d-1}(t)\right]P_i + t^dP_d\\
&= (1-t)^dP_0 + \sum_{i=1}^{d-1} \left[B_i^d(t)\right]P_i + t^dP_d\\
&= \sum_{i=0}^{d} B_i^d(t).
\end{aligned}$$

Proof of the Control Point Property for Polar forms

In the previous section we showed that a polar form can be evaluated with nested linear interpolation, and we did this using the affine and symmetry properties. If we also now use the substitution property, we can arrive at the control point property. For $d = 2$, we can set $u_1 = u_2 = t$ in the table:

$$\begin{array}{ccc}
F[0,0] & & \\
 & F[u_1,0] & \\
F[0,1] & & F[u_1,u_2]\\
 & F[u_1,1] & \\
F[1,1] & &
\end{array}$$

to obtain the new table:

$$\begin{array}{ccc}
F[0,0] & & \\
 & F[t,0] & \\
F[0,1] & & F[t,t]=\gamma(t)\\
 & F[t,1] & \\
F[1,1] & &
\end{array}$$

Note that in this table all nested linear interpolation is done with parameter t, just as in the case for the Bezier curve. Moreover, we arrive on the right at the function $\gamma(t)$. Since this table behaves identically to the Bezier point table:

$$\begin{array}{ccc} P_0^0 & & \\ & P_0^1 & \\ P_1^0 & & P_0^2 = \gamma(t) \\ & P_1^1 & \\ P_2^0 & & \end{array}$$

we would like to conclude that the starting values must be the same. To see that this must be true, we can use the labels:

$$Q_0 = F[0,0], Q_1 = F[0,1], \quad \text{and} \quad Q_2 = F[1,1].$$

Now suppose that at least one of the equalities $P_0 = Q_0$, $P_1 = Q_1$, and $P_2 = Q_2$ is false. Then we have two sets of control points which give the same result through nested linear interpolation:

$$\gamma_{[P_0,P_1,P_2]}(t) = \gamma_{[Q_0,Q_1,Q_2]}(t).$$

But from the previous section, we know that this is equivalent to the BB-forms being equal:

$$\sum_{i=0}^{d} B_i^d(t) P_i = \sum_{i=0}^{d} B_i^d(t) Q_i.$$

But then if $P_i = (a_i, b_i)$ and $Q_i = (c_i, d_i)$, we can focus on the first coordinate and obtain:

$$\sum_{i=0}^{d} a_i B_i^d(t) = \sum_{i=0}^{d} c_i B_i^d(t).$$

But since the Bernstein polynomials are a basis of P_d, any polynomial is uniquely represented by a choice of coefficients. So the only way for the last sums to be equal is if $a_i = c_i$ for each i. The same applies to the second coordinate, and so we must have $P_i = Q_i$ for all i, which shows that the control point property is true.

Proof of Uniqueness of Polar Forms

In order to show that there is only one polar form for any polynomial, or for any parametric polynomial curve, we use the facts from the previous section.

Suppose that a parametric polynomial curve $\gamma(t)$ has two polar forms $F[u_1, \ldots, u_d]$ and $G[u_1, \ldots, u_d]$. Each of these functions must then satisfy the three defining properties of a polar form. We have also shown in the previous section that each polar form is computable by nested linear interpolation from the special values, with 0's or 1's as arguments, which are in turn equal to the control points of $\gamma(t)$. But then F and G are both computing exactly the same output, and so are equal as functions.

Proof of Reparametrization Property of Polar Forms

The proof of the reparametrization property follows the same ideas used in the proof of the control point property.

For the reparametrization property we assume that we have a curve $\gamma(t)$ and also a reparametrization:

$$\alpha(t) = \gamma((1-t)a + tb),$$

for some constants a and b. We also assume that we have a polar form for $\gamma(t)$, called $F[u_1, \ldots, u_d]$. Assume again, for simplicity, that $d = 2$. Then we can construct a table for nested linear interpolation, starting from the values $F[a,a]$, $F[a,b]$, and $F[b,b]$. But first we work out a few linear interpolations:

$$(1-t)F[a,a] + tF[a,b] = F[a, (1-t)a + tb],$$

and also:

$$(1-t)F[a,b] + tF[b,b] = F[(1-t)a+tb, b] = F[b, (1-t)a+tb].$$

Combining these two we have:

$$\begin{aligned}
&(1-t)F[a,(1-t)a+tb] + tF[b,(1-t)a+tb] \\
&= F[(1-t)a+tb,(1-t)a+tb] \\
&= \gamma((1-t)a+tb) = \alpha(t).
\end{aligned}$$

The table then looks like:

$$\begin{array}{ccc}
F[a,a] & & \\
 & F[a,(1-t)a+tb] & \\
F[a,b] & & F[(1-t)a+tb,(1-t)a+tb] = \alpha(t) \\
 & F[b,(1-t)a+tb] & \\
F[b,b] & &
\end{array}$$

This shows that $\alpha(t)$ can be computed by nested linear interpolation from the values $F[a,a]$, $F[a,b]$ and $F[b,b]$. As before, we conclude that these must be the control points of $\alpha(t)$. This proves the reparametrization property for $d = 2$. The cases for higher d follow the same argument.