Lecture 18

Main Points:

- Tangent construction of conics
- Implicit forms for quadratic Bezier Curves

Tangent construction of conics

Suppose that in the five point construction we let the point P_1 approach P_0 along the line between them until they finally meet at the point P_0. If we could watch the conics smoothly deform as we perform this transition, we would see that the limit as P_1 approaches P_0 is in fact the conic which has tangent line at P_0 given by the initial line between P_0 and P_1.

We can do the same with the points P_2 and P_3, allowing P_3 to approach P_2, and obtaining a conic in the limit which has tangent line at P_2 given by the initial line through P_2 and P_3. In this process we can also note that the two lines $L_{0,2}$ and $L_{2,3}$ have become the same line, which we call simply L. We also relabel the lines $L_{0,2}$ to L_0 and $L_{2,3}$ to L_2, since these are now the tangent lines at P_0 and P_2. We then have:

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = 0.$$

We can solve for c just as we did in the five point construction, by inserting P_4 into the equation. The resulting equation will then pass through P_0, P_2, and P_4, and have tangent lines L_0 at P_0 and L_2 at P_2.

The above discussion gives an intuitive idea of how these constructions work. A full verification requires techniques in algebraic geometry, which we will not pursue here.

Implicit form of a quadratic Bezier curve

The quadratic Bezier curve with control points P_0, P_1, and P_2 has an equation of the form

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = 0,$$

where L_0 is the line containing P_0 and P_1, and is tangent to the curve at P_0, L_2 is the line containing P_1 and P_2, and is tangent to the curve at P_2, and L is the line containing P_0 and P_2.

To solve for c we can use a third point on the the curve, such as $\gamma(\frac{1}{2})$.

Examples:

- Find the implicit equation for the curve $\gamma(t)$ with control points $P_0 = (0,0)$, $P_1 = (1,0)$ and $P_2 = (2,4)$. We find the linear equations:

  $$L_0(x,y) = y = 0, \qquad L_2(x,y) = 4x - y - 4 = 0, \qquad \text{and} \qquad L(x,y) = y - 2x = 0,$$

  which gives:

  $$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = y(4x - y - 4) + c(y - 2x)^2 = 0.$$

  Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

  $$\begin{array}{lll} P_0 = (0,0) & & \\ & P_0^1 = (\frac{1}{2}, 0) & \\ P_1 = (1,0) & & P_0^2 = (1,1) = \gamma(\frac{1}{2}) \\ & P_1^1 = (\frac{3}{2}, 2) & \\ P_2 = (2,4) & & \end{array}$$

Finally, we plug in $(1,1)$ to $f_c(x,y)=0$ and solve for c:

$$\begin{aligned} f_c(1,1) &= 1(4-1-4)+c(1-2)^2 \\ &= -1+c \\ &= 0, \end{aligned}$$

which means that $c=1$, and the equation for $f_c = f$ is:

$$\begin{aligned} f(x,y) &= y(4x-y-4)+(y-2x)^2 \\ &= 4xy-y^2-4y+y^2-4xy+4x^2 \\ &= 4x^2-4y \\ &= 0, \end{aligned}$$

and this is equivalent to:

$$4x^2-4y=0,$$

which means that this Bezier curve has the simple implicit form:

$$y=x^2.$$

- We can also reverse this procedure and assume that a Bezier curve has implicit form $y=x^2$, and specify some of the control points and ask for the remaining control points. For example, we could leave $P_2=(2,4)$, and change P_0 to $P_0=(-1,1)$. How do we find P_1?

  Since P_1 determines the tangent line to the curve at P_0, because

  $$\gamma'(0)=2\mathbf{v}_1=2(P_1-P_0),$$

  and since P_1 also determines the tangent line to the curve at P_2, because

  $$\gamma'(1)=2\mathbf{v}_2=2(P_2-P_1),$$

  we see that P_1 must be the intersection point of the two tangent lines at P_0 and P_2. But the tangent lines are easily found using the implicit form. We find the derivative $y'=2x$, and thus the tangent slope at $(2,4)$ is $y'(2)=4$ and the tangent slope at $(-1,1)$ is $y'(-1)=-2$. Thus the tangent line at $(2,4)$ is:

  $$y=4(x-2)+4=4x-4,$$

  and the tangent line at $(-1,1)$ is:

  $$y=-2(x+1)+1=-2x-1.$$

  Then to find the intersection we set:

  $$4x-4=y=-2x-1,$$

  which means $6x=3$, or $x=\frac{1}{2}$, and $y=4\frac{1}{2}-4=-2$. So we have:

  $$P_1=\left(\frac{1}{2},-2\right).$$

  We can perform a small consistency check by computing the point $\gamma(\frac{1}{2})$ with these control points, which should be a point on the curve $y=x^2$. Again, we use the Bezier point array:

  $$\begin{array}{ccc} P_0=(-1,1) & & \\ & P_0^1=(-\frac{1}{4},-\frac{1}{2}) & \\ P_1=(\frac{1}{2},-2) & & P_0^2=(\frac{1}{2},\frac{1}{4})=\gamma(\frac{1}{2}) \\ & P_1^1=(\frac{5}{4},1) & \\ P_2=(2,4) & & \end{array}$$

  and indeed $\gamma(\frac{1}{2})=(\frac{1}{2},\frac{1}{4})$ is a point on $y=x^2$.

- Find the implicit equation for the curve $\gamma(t)$ with control points $P_0 = (0, 2)$, $P_1 = (0, 0)$ and $P_2 = (2, 0)$. We expect this one to be a parabola with axis of symmetry along the line $y = x$. This means that the equation should have a nonzero 'cross term' xy. We find the linear equations:

$$L_0(x, y) = x = 0, \qquad L_2(x, y) = y = 0, \qquad \text{and} \qquad L(x, y) = x + y - 2 = 0,$$

which gives:

$$f_c(x, y) = L_0(x, y)L_2(x, y) + cL(x, y)^2 = xy + c(x + y - 2)^2 = 0.$$

Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

$$\begin{array}{lll} P_0 = (0,2) & & \\ & P_0^1 = (0,1) & \\ P_1 = (0,0) & & P_0^2 = (\frac{1}{2}, \frac{1}{2}) = \gamma(\frac{1}{2}) \\ & P_1^1 = (1,0) & \\ P_2 = (2,0) & & \end{array}$$

Finally, we plug in $(\frac{1}{2}, \frac{1}{2})$ to $f_c(x, y) = 0$ and solve for c:

$$\begin{aligned} f_c\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{2} \cdot \frac{1}{2} + c\left(\frac{1}{2} + \frac{1}{2} - 2\right)^2 \\ &= \frac{1}{4} + c \\ &= 0, \end{aligned}$$

which means that $c = -\frac{1}{4}$, and the equation for $f_c = f$ is:

$$\begin{aligned} f(x, y) &= xy + c(x + y - 2)^2 \\ &= xy - \frac{1}{4}(x + y - 2)^2 \\ &= 0. \end{aligned}$$

This is equivalent to:

$$4xy - (x + y - 2)^2 = 0,$$

or

$$4xy - (x^2 + 2xy + y^2 - 4x - 4y + 4) = 0,$$

and in standard form we have:

$$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0.$$

Finally, we can check the discriminant $\Delta = B^2 - 4AC = (-2)^2 - 4 = 0$, which confirms that we have a parabola.