Lecture 21

Main Points:

- General discussion of B-splines and B-spline curves
- Order of continuity of B-splines based on knot sequence

Definition of B-splines of degree d for a knot sequence $\mathbf{t}$

Given a knot sequence $\mathbf{t}$ with $t_0 \leq t_1 \leq \ldots \leq t_N$, we define for any degree $d \geq 0$, and for $0 \leq i \leq N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d$$

Some points regarding B-splines:

- The set $\mathcal{B}_d(\mathbf{t})$ or $\mathcal{B}_{d,\mathbf{t}}$ is the set of B-splines associated to the knot sequence $\mathbf{t}$: $\mathcal{B}_d(\mathbf{t}) = \{\mathcal{B}_0^d(t), \ldots, \mathcal{B}_{N-d-1}^d(t)\}$
- The interval of support of a nonzero B-spline $\mathcal{B}_i^d(t)$ is the interval where it is nonzero: (t_i, t_{i+d+1}).
- Note: From the definition of B-spline, if $t_i = t_{i+d+1}$ (which means that all t_j with $i < j < i+d+1$ are also equal to t_i) then the B-spline $\mathcal{B}_i^d(t)$ is just zero for all t, since it starts with the factor $(t_{i+d+1} - t_i) = 0$.
- Positivity of nonzero B-splines: $\mathcal{B}_i^d(t) > 0$ for $t_i < t < t_{i+d+1}$.

Orders of continuity for a B-spline

The possible orders of continuity at a given breakpoint c in a spline space with order of continuity r are: $r, r+1, \ldots, d$ for $r = d-m$, where m is the multiplicity of the knots equal to c. Given the exact descriptions of bases, and the notes above, it is possible to say exactly which functions in a basis have which exact orders of continuity. We say that a function f has exact order of continuity r at c if $f^{(j)}(c)$ exists for $j = 0, \ldots r$, but $f^{(r+1)}(c)$ does not exist.

The exact order of continuity of a B-spline $\mathcal{B}_i^d(t)$ at each of the knot values $t_i, \ldots, t_{i+d+1}$ is given by $d - m(t_i)$ where $m(t_i)$ is the multiplicity of the value t_i in the *subsequence* $t_i, \ldots, t_{i+d+1}$. (Note: This is NOT necessarily the same as the multiplicity in the whole knot sequence $\mathbf{t}$.)

Examples:

- A degree $d = 0$ B-spline has two knot values, say t_i and t_{i+1}. As we saw before, if $t_i < t_{i+1}$, then $\mathcal{B}_i^0(t) = 1$, for $t_i \leq t < t_{i+1}$ and is zero otherwise. Since each of t_i and t_{i+1} has multiplicity one, this B-spline must have order of continuity $d - m = 0 - 1 = -1$ at each value, which means a *discontinuity.*
- A degree $d = 1$ B-spline has three knot values, say t_i, t_{i+1}, and t_{i+2}. We can guess the shape of such a B-spline depending on the multiplicities of the knot values. For example, if the multiplicities are all one, so that $t_i < t_{i+1} < t_{i+2}$, then the $\mathcal{B}_i^1(t)$, has order of continuity $d = m = 1 - 1 = 0$ at each knot values, which means that it is continuous there. Since it is also positive (by the property above) and piecewise linear for $t_i < t < t_{i+2}$, and equal to zero at t_i and t_{i+2}, we see that the shape of the function is a hat shape.
- Now suppose $d = 1$ and $t_i = t_{i+1} < t_{i+2}$. Then $\mathcal{B}_i^1(t)$ has order of continuity $d - m = 1 - 2 = -1$ at $t = t_i$ and order of continuity 0 at $t = t_{i+2}$.
- A degree $d = 2$ B-spline has four knot values t_i, t_{i+1}, t_{i+2}, and t_{i+3}. We can list some examples with orders of continuity and multiplicities:

degree	knot values	multiplicities	orders of continuity
2	0,1,2,3	1,1,1,1	1,1,1,1
2	0,0,1,2	2,1,1	0,1,1
2	0,1,1,2	1,2,1	1,0,1
2	0,1,2,2	1,1,2	1,1,0
2	0,2,2,2	1,3	1,-1

- With the knot sequence $\mathbf{t} = \{0, 1, 2, 2, 3, 3, 3, 4, 5, 6\}$, the spline $\mathcal{B}_1^2(t)$ is based on the subsequence $\{1, 2, 2, 3\}$. The order of continuity at 3 will be 1 since the multiplicity of 3 in this subsequence is 1 and $r = d - m = 2 - 1 = 1$.

  Here is a table of the B-splines associated to this knot sequence, with their subsequence, multiplicities and orders of continuity. Note: the index i of each B-spline determines the starting knot value t_i.

index i	B-spline	knot values	multiplicities	orders of continuity
0	$\mathcal{B}_0^2(t)$	0,1,2,2	1,1,2	1,1,0
1	$\mathcal{B}_1^2(t)$	1,2,2,3	1,2,1	1,0,1
2	$\mathcal{B}_2^2(t)$	2,2,3,3	2,2	0,0
3	$\mathcal{B}_3^2(t)$	2,3,3,3	1,3	1,-1
4	$\mathcal{B}_4^2(t)$	3,3,3,4	3,1	-1,1
5	$\mathcal{B}_5^2(t)$	3,3,4,5	2,1,1	0,1,1
6	$\mathcal{B}_6^2(t)$	3,4,5,6	1,1,1,1	1,1,1,1

  The graphs of each of the above B-splines can be drawn, up to a scaling factor, based on the orders of continuity at each of the knot values.

Curry-Schoenberg Theorem for B-spline bases.

The Curry-Schoeberg Theorem states that $P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$, with $\mathbf{r} = (r_1, \ldots, r_{k-1})$ and $\mathbf{m} = (m_1, \ldots, m_{k-1})$, has a basis consisting of B-splines associated to a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$ given by $t_0 \leq t_1 \leq \cdots \leq t_d \leq u_0$, and $u_k \leq t_{N-d} \leq \cdots \leq t_N$. The middle part of the knot sequence $t_{d+1}, \ldots, t_{N-d-1}$ corresponds exactly to the sequence of breakpoints $u_1, \ldots, u_1, u_2, \ldots, u_2, \quad \ldots \quad, u_{k-1}, \ldots, u_{k-1}$ where the multiplicity of each u_i is $m_i = d - r_i$.

B-spline curves

A B-spline curve can be written in the form:

$$\gamma(t) = \sum_{i=0}^{N-d-1} \mathcal{B}_i^d(t) P_i,$$

where the points P_i are called the control points. This form is analogous to the BB-form form Bezier curves, where the functions of t are the Bernstein polynomials, and the index runs from $i = 0$ to d since P_d has dimension $d+1$. In the case of the B-splines, the dimension of the vector space of splines has dimension $N - d$. By the Curry-Schoenberg Theorem, this number aligns with the dimension of the vector space $P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$, for some orders of continuity $(r_1, \ldots, r_{k-1})$, which is $d + 1 + \sum_{i=1}^{k-1} d - r_i$. So we have:

$$dim\left(P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]\right) = d + 1 + \sum_{i=1}^{k-1} d - r_i = N - d.$$

Since there are many different bases for this vector space of splines, the curve $\gamma(t)$ can be written in different forms without changing the function. This particular form also shares the property that control points affect the shape of the curve, just as with the BB-form, and that the curve can be evaluated by nested linear interpolation. The NLI algorithm for evaluating B-spline curves is called the DeBoor Algorithm.