Lecture 23

Main Points:

- Writing B-splines in terms of shifted power functions
- Orders of continuity for sums of shifted power functions
- Orders of continuity for B-splines

Writing a B-spline in terms of shifted power functions

To write a B-spline as a sum of shifted power functions, we use the definition of the divided difference. This says that, for instance, the divided difference

$$[t_i, \ldots, t_{i+d+1}]g$$

is equal to the coefficient of t^d in the interpolating polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_d t^d,$$

which matches the function g at the data values $t_i, \ldots, t_{i+d+1}$. This definition allows us to compute this divided difference by solving for a_d by any method. One method was to use the recursion formula for divided differences. Another method is Cramer's Rule.

Recall Cramer's Rule for a linear system with variables x_1, x_2, and x_3, and augmented matrix:

$$\left(\begin{array}{ccc|c} a_{1,1} & a_{1,2} & a_{1,3} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & b_3 \end{array}\right)$$

This system can also be written:

$$A\mathbf{x} = \mathbf{b}.$$

If $\det(A) \neq 0$ then the solution can be given by Cramer's Rule as:

$$x_1 = \frac{1}{\det(A)} \begin{vmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{vmatrix}, \qquad x_2 = \frac{1}{\det(A)} \begin{vmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{vmatrix}, \qquad x_3 = \frac{1}{\det(A)} \begin{vmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ a_{3,1} & a_{3,2} & b_3 \end{vmatrix}.$$

We can apply this to solve for the coefficients of an interpolating polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

We have changed the variable to be x since our application to the definition of B-splines uses x as the dummy variable. If we suppose that the interpolating polynomial fits the data values $x = t_0$, $x = t_1$, and $x = t_2$ for a function $g(x)$, with $t_0 < t - 1 < t_2$, then the linear system, with variables a_0, a_1, and a_2 becomes:

$$\left(\begin{array}{ccc|c} 1 & t_0 & t_0^2 & g(t_0) \\ 1 & t_1 & t_1^2 & g(t_1) \\ 1 & t_2 & t_2^2 & g(t_2) \end{array}\right)$$

If we call the determinant of the coefficient matrix

$$D = \begin{vmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{vmatrix} = (t_1 - t_0)(t_2 - t_0)(t_2 - t_1),$$

then by Cramer's Rule, the solution for the variables is:

$$a_0 = \frac{1}{D}\begin{vmatrix} g(t_0) & t_0 & t_0^2 \\ g(t_1) & t_1 & t_1^2 \\ g(t_2) & t_2 & t_2^2 \end{vmatrix}, \qquad a_1 = \frac{1}{D}\begin{vmatrix} 1 & g(t_0) & t_0^2 \\ 1 & g(t_1) & t_1^2 \\ 1 & g(t_2) & t_2^2 \end{vmatrix}, \qquad a_2 = \frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_1 & g(t_2) \end{vmatrix}.$$

So we see that the divided difference for degree $d = 2$ can be written:

$$[t_0, t_1, t_2]g = a_2 = \frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_2 & g(t_2) \end{vmatrix}.$$

Finally, we can use this to write a degree $d = 1$ B-spline as a sum of shifted power functions, with $g(x) = (t - x)_+^1$:

$$\begin{aligned}
\mathcal{B}_0^1(t) &= (-1)^2(t_2 - t_0)[t_0, t_1, t_2](t - x)_+^1 \\
&= (t_2 - t_0)\frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_2 & g(t_2) \end{vmatrix} \\
&= \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_0)(t_2 - t_1)}\left[g(t_0)\begin{vmatrix} 1 & t_1 \\ 1 & t_2 \end{vmatrix} - g(t_1)\begin{vmatrix} 1 & t_0 \\ 1 & t_2 \end{vmatrix} + g(t_2)\begin{vmatrix} 1 & t_0 \\ 1 & t_1 \end{vmatrix}\right] \\
&= \frac{1}{(t_1 - t_0)(t_2 - t_1)}\left[(t_2 - t_1)g(t_0) - (t_2 - t_0)g(t_1) + (t_1 - t_0)g(t_2)\right] \\
&= \frac{1}{(t_1 - t_0)}g(t_0) - \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_1)}g(t_1) + \frac{1}{(t_2 - t_1)}g(t_2) \\
&= \frac{1}{(t_1 - t_0)}(t - t_0)_+^1 - \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_1)}(t - t_1)_+^1 + \frac{1}{(t_2 - t_1)}(t - t_2)_+^1
\end{aligned}$$

We can verify that this last expression has the shape of the hat function, by checking that the value at $t = t_0$ is zero, and at $t = t_1$ is 1, and at $t = t_2$ is zero. Also, it is clear that for $t < t_0$ the value is zero. To check that the function is also zero for $t > t_2$, we can use a property of determinants. Recall that a determinant is linear as a function of any single row or column. Since the functions lose their piecewise nature for $t > t_2$, the determinant becomes:

$$\begin{vmatrix} 1 & t_0 & t - t_0 \\ 1 & t_1 & t - t_1 \\ 1 & t_2 & t - t_2 \end{vmatrix} = \begin{vmatrix} 1 & t_0 & t \\ 1 & t_1 & t \\ 1 & t_2 & t \end{vmatrix} - \begin{vmatrix} 1 & t_0 & t_0 \\ 1 & t_1 & t_1 \\ 1 & t_2 & t_2 \end{vmatrix} = t\begin{vmatrix} 1 & t_0 & 1 \\ 1 & t_1 & 1 \\ 1 & t_2 & 1 \end{vmatrix} - 0 = 0 - 0 = 0.$$

Examples:

- Suppose that t_0, t_1, and t_2 are consecutive integers, such as $1, 2, 3$. Then the function $\mathcal{B}_0^1(t)$ can be written as:

$$\mathcal{B}_0^1(t) = (t - 1)_+^1 - 2(t - 2)_+^1 + (t - 3)_+^1.$$

Next, we do the same for a degree 2 B-spline $\mathcal{B}_i^2(t)$, with simple knot values $t_i = a$, $t_{i+1} = b$, $t_{i+2} = c$, $t_{i+3} = d$, and $a < b < c < d$.

As in the previous case, $[a,b,c,d]g$, with $g(x) = (t-x)_+^2$, can be calculated as the coefficient a_3 in the interpolating polynomial $p(x)$ matching g for the data values a, b, c, d, where $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. By Cramer's Rule with D given by

$$D = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

we have:

$$[a,b,c,d]g(x) = a_3 = \frac{1}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 1 & b & b^2 & g(b) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix}.$$

So if $t_i = a$, $t_{i+1} = b$, $t_{i+2} = c$, $t_{i+3} = d$, then we have

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(d-a)[a,b,c,d](t-x)_+^2 \\
&= \frac{-(d-a)}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 1 & b & b^2 & g(b) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} \\
&= \frac{-(d-a)}{D}\left[-g(a)\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + g(b)\begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - g(c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & d & d^2 \end{vmatrix} + g(d)\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix}\right] \\
&= (d-a)\left[\frac{g(a)}{D}\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - \frac{g(b)}{D}\begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + \frac{g(c)}{D}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & d & d^2 \end{vmatrix} - \frac{g(d)}{D}\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix}\right] \\
&= (d-a)\left[\frac{g(a)}{(b-a)(c-a)(d-a)} - \frac{g(b)}{(b-a)(c-b)(d-b)} + \frac{g(c)}{(c-a)(c-b)(d-c)} - \frac{g(d)}{(d-a)(d-b)(d-c)}\right] \\
&= \frac{g(a)}{(b-a)(c-a)} - \frac{(d-a)g(b)}{(b-a)(c-b)(d-b)} + \frac{(d-a)g(c)}{(c-a)(c-b)(d-c)} - \frac{g(d)}{(d-b)(d-c)} \\
&= \frac{(t-a)_+^2}{(b-a)(c-a)} - \frac{(d-a)(t-b)_+^2}{(b-a)(c-b)(d-b)} + \frac{(d-a)(t-c)_+^2}{(c-a)(c-b)(d-c)} - \frac{(t-d)_+^2}{(d-b)(d-c)}
\end{aligned}$$

Examples:

- Let $a = 1$, $b = 2$, $c = 3$, $d = 4$. Then we have:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(4-1)[1,2,3,4](t-x)_+^2 \\
&= \frac{1}{2}(t-1)_+^2 - \frac{3}{2}(t-2)_+^2 + \frac{3}{2}(t-3)_+^2 - \frac{1}{2}(t-4)_+^2.
\end{aligned}$$

Now consider the case of non-simple knot sequence. For instance, we could let $t_i = t_{i+1} = a$, $t_{i+2} = c$, and $t_{i+3} = d$. Then we can compute $[a, a, c, d]g$, for $g(x) = (t - x)_+^2$, again using Cramer's Rule with D given by

$$D = D(aacd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (c-a)^2(d-a)^2(d-c).$$

Then we have:

$$[a, a, c, d]g = a_3 = \frac{1}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 0 & 1 & 2a & g'(a) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} = \frac{1}{D}\begin{vmatrix} 1 & a & a^2 & (t-a)_+^2 \\ 0 & 1 & 2a & 2(t-a)_+^1 \\ 1 & c & c^2 & (t-c)_+^2 \\ 1 & d & d^2 & (t-d)_+^2 \end{vmatrix}.$$

Applying this to the B-spline $\mathcal{B}_i^2(t)$, with knot sequence a, a, c, d, we get:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(d-a)[a, a, c, d](t-x)_+^2 \\
&= -\frac{d-a}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 0 & 1 & 2a & g'(a) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} \\
&= \frac{d-a}{D}\left[g(a)\begin{vmatrix} 0 & 1 & 2a \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - g'(a)\begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + g(c)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & d & d^2 \end{vmatrix} - g(d)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & c & c^2 \end{vmatrix}\right] \qquad (1) \\
&= \frac{d-a}{D}\left[g(a)\left[-(d^2-c^2) + 2a(d-c)\right] - g'(a)(c-a)(d-a)(d-c) + g(c)(d-a)^2 - g(d)(c-a)^2\right] \\
&= \frac{d-a}{D}\left[\left[-(d^2-c^2) + 2a(d-c)\right](t-a)_+^2 - (c-a)(d-a)(d-c)2(t-a)_+^1 + (d-a)^2(t-c)_+^2 - (c-a)^2(t-d)_+^2\right].
\end{aligned}$$

Note: In line (1) above, the determinants (which become the coefficients of the shifted power functions) can sometimes be recognized as Vandermonde or Confluent Vandermonde. In this line, the second one is Vandermonde, and the last two are Confluent Vandermonde, but the first one is neither of these types, so it is simply evaluated by the cofactor expansion formula for determinants.

Examples:

- Let $a = 1$, $c = 2$, $d = 3$. Then we have: $D = 4$, and we can write the B-spline:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(3-1)[1, 1, 2, 3](t-x)_+^2 \\
&= \frac{1}{2}\left[-3(t-1)_+^2 - 4(t-1)_+^1 + 4(t-2)_+^2 - (t-3)_+^2\right].
\end{aligned}$$

Lowest degree shifted power function in a B-spline has non-zero coefficient

The lowest degree shifted power function in a B-spline expansion, given by the above determinant formula coming from Cramer's Rule, must have non-zero coefficient. The reason for this is simply that the coefficient comes from a determinant which is either Vandermonde or Confluent Vandermonde. This can be seen by deleting the row corresponding to the lowest degree shifted power function, say $a(t-t_j)_+^k$, which must be the highest derivative of some shifted power function $(t-t_j)_+^n$. The remaining rows corresponding to t_j have all the lower order derivatives, and thus the determinant must be Confluent Vandermonde (or regular Vandermonde, which is of course a sub-case).

Orders of continuity for sums of shifted power functions

We have seen that the shifted power function $(t-c)_+^k$ is continuous to all orders at all points not equal to c, and can be seen to have exact order of continuity $k-1$ at $t=c$.

It easy to extend this fact to sums of such functions. In particular, if $f(t)$ is a sum of shifted power functions, then the order of continuity of f is simply the minimum of all orders of continuity of the summands. If the function $(t-c)_+^k$ is the one that achieves this minimum, then clearly that lowest order of continuity is $k-1$ and it occurs for the value $t=c$.

Examples:

- The function
$$f(t) = 7(t-4)_+^4 + 3(t-4)_+^6 - 5(t-7)_+^5$$
has exact order of continuity 3 which is attained by the summand $7(t-4)_+^4$ at the value $t=4$.
- The function
$$f(t) = 3(t-4)_+^3 + 2(t-4)_+^4 - 5(t-7)_+^5 + 6(t-8)_+^4 - (t-8)_+^7$$
has exact order of continuity 0 which is attained by the summand $2(t-3)_+^1$ at the value $t=3$.

Further Details on orders of continuity for sums of shifted power functions

Recall that a function $f(t)$ has order of continuity r at $t=c$ if f is continuous at $t=c$ and each of the derivative functions $f', f'', \ldots, f^{(r)}$ are continuous at $t=c$. If, in addition, the function $f^{(r+1)}$ is *not* continuous at $t=c$, then we say that f has *exact order of continuity* r at $t=c$. If f and all of its derivatives are continuous at $t=c$ then we say f has infinite order of continuity, or simply f is continuous to all orders at $t=c$.

If we let $f(t)=(t-c)_+^k$ then the derivatives of f are:

$$f'(t) = k(t-c)_+^{k-1}, f''(t) = k(k-1)(t-c)_+^{k-2}, \ldots, f^{(k-1)}(t) = k!(t-c)_+^1.$$

Note: The function $f(t)=(t-c)_+^1$ is not differentiable at $t=c$, although it is continuous there. The function $f(t)=(t-c)_+^0$ is neither continuous nor differentiable at $t=c$.

Let $f(t)$ be defined as a sum:

$$f(t) = a_1(t-u)_+^{j_1} + a_2(t-u)_+^{j_2} + \cdots + a_n(t-u)_+^{j_n},$$

with all $a_i \neq 0$, and $j_1 < j_2 < \cdots < j_n$. Then the exact order of continuity of f at u is simply $j_1 - 1$. This follows from the above, since the higher powers are differentiable to higher orders.

Now let $f(t)$ be defined as a sum:

$$f(t) = a_1(t-u_1)_+^{j_1} + a_2(t-u_2)_+^{j_2} + \cdots + a_n(t-u_n)_+^{j_n},$$

with all $a_i \neq 0$, and $u_1 < u_2 < \cdots < u_n$. Then the exact order of continuity of f at u_i is simply $j_i - 1$. It is then a fact that f is a member of any vector space of piecewise polynomial functions of the form:

$$f \in P_{d,\mathbf{r}}^{n+1}[u_0, u_1, \ldots, u_n, u_{n+1}],$$

where $u_0 < u_1$ and $u_{n+1} > u_n$, and d is the maximum of the j_i, $i = 1, \dots, n$, and $\mathbf{r}$ is the vector of continuity conditions:

$$\mathbf{r} = \{r_1, r_2, \dots, r_n\}, \quad \text{with} \quad r_i = j_i - 1.$$

Finally, suppose we mix the above two cases by adding higher degree shifted power functions at each u_i, and now define f as:

$$\begin{aligned} f(t) &= a_{1,1}(t-u_1)_+^{j_{1,1}} + a_{1,2}(t-u_1)_+^{j_{1,2}} + \cdots + a_{1,n_1}(t-u_1)_+^{j_{1,n_1}} \\ &+ a_{2,1}(t-u_2)_+^{j_{2,1}} + a_{2,2}(t-u_2)_+^{j_{2,2}} + \cdots + a_{2,n_2}(t-u_2)_+^{j_{2,n_2}} \\ &\vdots \\ &+ a_{k,1}(t-u_k)_+^{j_{k,1}} + a_{k,2}(t-u_k)_+^{j_{k,2}} + \cdots + a_{k,n_k}(t-u_k)_+^{j_{k,n_k}} \end{aligned}$$

Then if each row is written with increasing powers $j_{i,1} < j_{i,2} < \cdots < j_{i,n_i}$, we can choose a minimal value $j_{m,1}$ from the first column and we have the order of continuity of f is $j_{m,1} - 1$.

Proof of orders of continuity of a B-spline

We can prove the claim that the order of continuity of a B-spline $\mathcal{B}_i^d(t)$ at t_j is given by the multiplicity of t_j in the subsequence $t_i, \dots, t_{i+d+1}$.

This fact follows from two previous statements above. First, the coefficient of the lowest degree shifted power function of type $(t-t_j)_+^k$ in the expansion of B-spline $\mathcal{B}_i^d(t)$ must be nonzero, for t_j in the sequence $t_i, \dots, t_{i+d+1}$. Then, by the above, the order of continuity of $\mathcal{B}_i^d(t)$ at t_j must be $d - m$, where m is the multiplicity of t_j in the sequence $t_i, \dots, t_{i+d+1}$.