Lecture 24

Main Points:

- Equivalence of DeBoor Algorithm and B-spline summation

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Let $\gamma(t)$ be a degree d B-spline curve defined by:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t)$$

for some knot sequence $\mathbf{t} = \{t_0, t_1, \ldots, t_N\}$ and control points $P_0, P_1, \ldots, P_{N-d-1}$, and t-values in the interval $[t_d, t_{N-d})$.

For the same knot sequence, control points and degree d, we can define, with the DeBoor algorithm, the point $P_J^{[d]}$ obtained through nested linear interpolation.

We will show that these two methods produce the same point:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) = P_J^{[d]}.$$

Some ingredients in the proof:

The B-spline recursion formula

The (DeBoor-Cox) recursion formula for B-splines of degree d associated to a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$ is:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t)$$

The DeBoor Algorithm

The nested linear interpolation formula for B-splines is called the DeBoor Algorithm. For a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$, and control points $P_0, \ldots, P_{N-d-1}$, and $t \in [t_d, t_{N-d})$, we first define the index J to be the unique value such that $t \in [t_J, t_{J+1})$. Then the DeBoor points can be computed according to the nested linear interpolation scheme given by

$$P_i^{[k]} = \frac{t_{i+d-(k-1)} - t}{t_{i+d-(k-1)} - t_i} P_{i-1}^{[k-1]} + \frac{t - t_i}{t_{i+d-(k-1)} - t_i} P_i^{[k-1]},$$

where $k = 1, \ldots, d$ and $i = J - d + k, \ldots, J$ and the final point is $P_J^{[d]}$. Since this process produces a point for each value $t \in [t_d, t_{N-d})$, we can call it a curve $\gamma(t)$ and write

$$\gamma(t) = P_J^{[d]}.$$

Proof of the equivalence of the B-spline summation and the DeBoor Algorithm

Define $\gamma(t)$ as above according to the B-spline summation formula and also the DeBoor algorithm. Then we claim:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) = P_J^{[d]}.$$

Proof:

The first step is to note that some of the B-splines in the the above summation are zero for $t \in [t_J, t_{J+1})$. In particular, each B-spline $\mathcal{B}_i^d(t)$ has support equal to the interval (t_i, t_{i+d+1}) and thus $\mathcal{B}_i^d(t) = 0$ if $i + d + 1 \leq J$, or if $i \geq J + 1$. This leaves only the B-splines with indices $J - d, \ldots, J$.

Below, we make this adjustment to the summation indices, then insert the B-spline recursive form and adjust the index i in the second term in order to recombine terms with the same B-spline.

$$\begin{aligned}
\gamma(t) &= \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d}^{J} P_i \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d}^{J} P_i \left[\frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t) \right] \\
&= \sum_{i=J-d}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d}^{J} P_i \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t) \\
&= \sum_{i=J-d}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d+1}^{J+1} P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t)
\end{aligned}$$

Now, since $t \in [t_J, t_{J+1})$, we have $\mathcal{B}_{J+1}^{d-1}(t) = 0$ and $\mathcal{B}_{J-d}^{d-1}(t) = 0$, which allows us to eliminate the first term in the first summation, and the last term in the second summation, in the above line. Continuing, we can now recognize the affine sum of points coming from the DeBoor algorithm, and we have:

$$\begin{aligned}
\gamma(t) &= \sum_{i=J-d+1}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d+1}^{J} P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} \left[P_i \frac{t - t_i}{t_{i+d} - t_i} \ + \ P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \right] \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} \left[\frac{t - t_i}{t_{i+d} - t_i} P_i^{[0]} \ + \ \frac{t_{i+d} - t}{t_{i+d} - t_i} P_{i-1}^{[0]} \right] \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} P_i^{[1]} \mathcal{B}_i^{d-1}(t).
\end{aligned}$$

We can now apply the entire above process to this summation recursively to get:

$$
\begin{aligned}
\gamma(t) &= \sum_{i=J-d}^{J} P_i^{[0]} \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d+1}^{J} P_i^{[1]} \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+2}^{J} P_i^{[2]} \mathcal{B}_i^{d-2}(t) \\
&\vdots \\
&= \sum_{i=J}^{J} P_J^{[d]} \quad \mathcal{B}_d^0(t) \\
&= P_J^{[d]}.
\end{aligned}
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