Lecture 26

Main Points:

- Interpolation with B-splines
- Schoenberg-Whitney Theorem

General Interpolation with a vector space of functions

If we have a vector space V of functions, with dimension n, then we can ask questions about interpolation using functions in V.

For instance, given data values $s_1, \ldots, s_n$ and some data function $g(t)$, can we find a function $f(t) \in V$ such that:

$$f(s_i) = g(s_i), \qquad i = 1, \ldots, n?$$

Note: This question is independent of the basis chosen for V. This means that if the interpolation problem has a solution, then we can write the function $f(t)$ in any basis that we like.

We know that in the case that $V = P_d$, and thus $n = d+1$, we have the existence and uniqueness of the interpolating polynomial, which says that this interpolation problem always has a unique solution.

We will see that this property does *not* always hold in other vector spaces of functions, in particular for splines.

Interpolation with splines

If we have a vector space of polynomial spline functions such as

$$V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$$

with $\mathbf{r} = (r_1, \ldots, r_k)$, and dimension

$$n = d + 1 \sum_{i=1}^{k-1} d - r_i$$

then we can ask questions about interpolation using functions in V.

For instance, given data values $s_1, \ldots, s_n$ and some data function $g(t)$, can we find a function $f(t) \in V$ such that:

$$f(s_i) = g(s_i), \qquad i = 1, \ldots, n?$$

The problem that comes up immediately is that since spline functions are 'locally polynomial', we cannot put 'too many' of the data values 'too close' together.

More precisely, if we put $d+2$ or more data values in the interval $[u_i, u_{i+1}]$, then we cannot expect to find a solution $f(t)$. This is because on the interval $[u_i, u_{i+1}]$ the function $f(t)$ is given by a polynomial $p(t)$ in P_d which must have degree at most d. But we cannot match such a polynomial to g at more than $d+1$ points, except if we have the coincidence that all the points lie on a polynomial of degree at most $d+1$ already.

So, a necessary condition to have a solution f to the above interpolation problem is that: Not more than $d+1$ of the data values lie in any interval of the form $[u_i, u_{i+1}]$.

Examples:

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 0.8 & s_{10} = 3.8 \end{array}$$

  Since there are five data values in the interval $[0,1]$, and the dimension of the space of cubic polynomials is only four, we know that we cannot choose an arbitrary data function $g(t)$ and expect to find a unique solution to the interpolation problem in V. In other words, given $g(t)$, we cannot expect to find a solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$. Of course, it is always possible that there could exist a solution simply by chance, or by design, for instance if we chose g to be in V. But in general the interpolation problem will not have a solution at all.

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 1.2 & s_{10} = 3.8 \end{array}$$

  Now we see that there are at most four data values in each of the sub-intervals, so at least there is a chance that when we choose an arbitrary data function $g(t)$ we might be able to find a solution to the interpolation problem in V. In other words, given $g(t)$, we might be able to find a solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$.

The Schoenberg-Whitney Theorem gives a precise criterion for the existence and uniqueness of a solution to the interpolation problem for spline vector spaces.

Schoenberg-Whitney Theorem

Let $\mathbf{t} = \{t_0, \ldots, t_N\}$ be a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$, with $dim(V) = N - d = n$. Now suppose we have a sequence of data values $u_0 \le s_0 < s_1 < \cdots < s_{n-1} \le u_k$, and a data function $g(t)$. Then there exists a unique function $f(t) \in V$, satisfying $f(s_i) = g(s_i)$, for $i = 0, \ldots, n-1$ if and only if:

$$\mathcal{B}^d_i(s_i) > 0, \qquad i = 0, \ldots, n-1.$$

Note: This last condition is equivalent to $t_i < s_i < t_{i+d+1}$ for the case of continuous B-splines, and $t_i \le s_i < t_{i+d+1}$ for a discontinuous B-spline. So, we can only have $s_i = t_i$ when the knot t_i has multiplicity $m = d + 1$ for the B-spline $\mathcal{B}^d_i(t)$.

Examples:

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 1.2 & s_{10} = 3.8 \end{array}$$

  Now let $\mathbf{t} = \{0,0,0,0,1,1,2,2,3,3,4,4,4,4\}$ be a knot sequence such that the B-splines of degree 3 for $\mathbf{t}$ are a basis of V. Then we can check the inequalities in the Schoenberg-Whitney Theorem:

$0 < 0.0 < 1$	$1 < 1.5 < 3$
$0 < 0.5 < 1$	$2 < 2.5 < 4$
$0 < 0.6 < 2$	$2 < 3.5 < 4$
$0 < 0.7 < 2$	$3 < 3.7 < 4$
$1 < 1.2 < 3$	$3 < 3.8 < 4$

Thus we can conclude that given any data function $g(t)$, there exists a unique solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$.

Proof of the Schoenberg-Whitney Theorem

Suppose that $\mathbf{t} = \{t_0, \ldots, t_N\}$ is a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$, with $dim(V) = N - d = n$. Also suppose we have a sequence of data values $u_0 \leq s_0 < s_1 < \cdots < s_{n-1} \leq u_k$, and a data function $g(t)$.

We need to show that there exists a unique function $f(t) \in V$, satisfying $f(s_i) = g(s_i)$, for $i = 0, \ldots, n-1$ if and only if:

$$\mathcal{B}^d_i(s_i) > 0, \quad i = 0, \ldots, n-1.$$

First we define the *collocation matrix* A to be the coefficient matrix of the linear system used to solve for f in the B-spline basis. In particular, we let

$$f(t) = c_0\mathcal{B}^d_0(t) + \cdots + c_n\mathcal{B}^d_{n-1}(t),$$

and then substitute $t = s_i$ into this equation to get the linear system

$$A\mathbf{x} = \mathbf{b}.$$

The general entry of A is then:

$$a_{i,j} = \mathcal{B}^d_j(s_i).$$

Showing that f exists and is unique in V, satisfying $f(s_i) = g(s_i)$, for $i = 0, \ldots, n-1$, is then equivalent to showing that A is nonsingular.

To show the forward implication of the theorem, we can prove the contrapositive. So suppose that $\mathcal{B}^d_m(s_m) = 0$ for some m. This is equivalent to saying that either $s_m \leq t_m$ or $s_m \geq t_{m+d+1}$. In the first case we have $\mathcal{B}^d_k(s_m) = 0$ for $k > m$, and the same holds in the second case for $k < m$. In the first case we also have $\mathcal{B}^d_j(s_i) = 0$ for $0 < i \leq m$ and $m \leq j \leq n-1$. In the second case we also have $\mathcal{B}^d_j(s_i) = 0$ for $m \leq i \leq n-1$ and $0 \leq j \leq m$. In either case, we have enough zeros in the entries of A to force $det(A) = 0$, since the determinant is a sum of signed products

$$\pm a_{0,\sigma_0}a_{1,\sigma_1}\cdots a_{n-1,\sigma_{n-1}}.$$

This proves the forward implication.

To prove the backward implication we suppose that the Schoenberg-Whitney condition is true:

$$\mathcal{B}^d_i(s_i) > 0, \quad i = 0, \ldots, n-1.$$

We need to show that $det(A) \neq 0$.

To show this, we need to look at the *variation-diminishing* property of B-splines.

If $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a sequence of numbers, denote the number of sign changes in $\mathbf{x}$ by $S^-\mathbf{x}$. Precisely, $S^-\mathbf{x}$ is the largest integer r with the property that for some set of indices j_k: $1 \leq j_1 < j_2 < \cdots < j_{r+1} \leq n$, it is true that $x_{j_i}x_{j_{i+1}} < 0$ for $i = 1, \ldots, r$.

The remaining part of the proof follows from the collocation matrix being positive definite.

Examples:

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$
\begin{array}{ll}
s_1 = 0.0 & s_6 = 1.5 \\
s_2 = 0.5 & s_7 = 2.5 \\
s_3 = 0.6 & s_8 = 3.5 \\
s_4 = 0.7 & s_9 = 3.7 \\
s_5 = 1.2 & s_{10} = 3.8
\end{array}
$$

  Now let $\mathbf{t} = \{0,0,0,0,1,1,2,2,3,3,4,4,4,4\}$ be a knot sequence such that the B-splines of degree 3 for $\mathbf{t}$ are a basis of V. Then we can check the inequalities in the Schoenberg-Whitney Theorem:

$$
\begin{array}{ll}
0 < 0.0 < 1 & 1 < 1.5 < 3 \\
0 < 0.5 < 1 & 2 < 2.5 < 4 \\
0 < 0.6 < 2 & 2 < 3.5 < 4 \\
0 < 0.7 < 2 & 3 < 3.7 < 4 \\
1 < 1.2 < 3 & 3 < 3.8 < 4
\end{array}
$$

  Thus we can conclude that given any data function $g(t)$, there exists a unique solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$.

Other Properties of B-splines

- **Partition of Unity**

  Let $\mathbf{t} = \{t_0, \ldots, t_N\}$ be a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$. Then for $t \in [u_0, u_k]$ we have:

$$\sum_{i=0}^{N-d-1} \mathcal{B}^d_i(t) = 1.$$

- **Bernstein knot sequence**

  Let $\mathbf{t} = \{0, 0, \ldots, 0, 1, 1, \ldots, 1\} = \{t_0, \ldots, t_{2(d+1)}\}$ be a knot sequence consisting of $d+1$ zeros followed by $d+1$ ones. Then the B-splines associated to this knot sequence $\mathbf{t}$, restricted to the interval $[0,1]$, are precisely the Bernstein polynomials.

- **Derivative of a B-spline function**

  The following derivative is similar to the derivative of the Bernstein polynomials:

$$\frac{d}{dt}\mathcal{B}^d_i(t) = d\left[\frac{\mathcal{B}^{d-1}_i(t)}{t_{i+d} - t_i} - \frac{\mathcal{B}^{d-1}_{i+1}(t)}{t_{i+d+1} - t_{i+1}}\right].$$