Lecture 27

Main Points:

- Introduction to Surfaces
- Review for Final

Total degree polynomial surfaces

Let $V = P_{d,\{x,y\}}$ denote the vector space of polynomials in the variables x and y with total degree at most d. A basis of V is:

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots, x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}.$$

The dimension of V is

$$dim(V) = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}.$$

Tensor Product Surfaces

The tensor product of two vector spaces of functions V and W, with bases $B_V = \{f_1, \ldots, f_n\}$ and $B_W = \{g_1, \ldots, g_m\}$ respectively, is defined as the vector space $V \otimes W$ with basis $\{h_{i,j} = f_i g_j : 1 \le i \le n, 1 \le j \le m\}$, where the basis functions $h_{i,j}(x, y) = f_i(x)g_j(y)$. Any function h in $V \otimes W$ can be expressed a sum:

$$h = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j}.$$

So, the definition of tensor product space is:

$$V \otimes W = \{\sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j} \ : \ h_{i,j}(x, y) = f_i(x)g_j(y), f_i \in B_V, g_j \in B_W\}.$$

The dimension of a tensor product space is:

$$dim(V \otimes W) = m \cdot n = dim(V) \cdot dim(W).$$

A tensor product surface is the graph of $z = h(x, y)$ with $h \in V \otimes W$.

The space $P_d \otimes P_d$ is a subspace of $P_{2d,\{x,y\}}$.

Similarly, the space $P_m \otimes P_n$ is a subspace of $P_{m+n,\{x,y\}}$.

Examples:

- Let $V = P_2$ with variable x, and let $W = P_3$ with variable y. Then the tensor product $V \otimes W$ has basis:

  $$\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3\}.$$

  This is a subset of the basis of of $P_{5,\{x,y\}}$:

  $$\{1, x, x^2, x^3, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5\}.$$

Points to be able to work out for the Final Exam:

- Find the polar form of a polynomial in Standard basis
- Find the polar form of a polynomial in Bernstein, or other, bases
- Identify the three defining properties of polar forms
- Verify if a given function satisfies one or more of the defining properties of polar forms
- Write the polar form for a parametric polynomial curve from the standard form
- Write the polar form for a Bezier curve from the BB-form
- Apply the control point property to find control points of a parametric polynomial curve in standard basis form
- Apply the reparametrization property to find control points of a reparametrized curve using the polar form of the original
- Use Nested Linear Interpolation to evaluate a polar form
- Verify equivalence of BB-form and NLI-form
- Write a Bezier curve in Cumulative form
- Find the derivative of a Bezier curve using Cumulative form
- Recognize a degenerate Bezier curve
- Classify the type of an implicit quadratic by using the Discriminant
- Use the five point construction to find a quadratic equation
- Use the tangent construction to find an implicit equation of a Bezier curve
- Use the tangent line properties of quadratic Bezier curves to solve for one control point given the other two
- Write the standard basis for a vector space of splines with continuity and multiplicity vectors
- Use a knot sequence to write a shifted power basis
- Work out a B-spline from the divided difference formula
- Write the knot subsequence for a B-spline based on a knot sequence and starting index
- Find the support of a B-spline (where it is nonzero)
- Find the exact order of continuity of a B-spline at one of its knot values using the multiplicity
- Use the Curry-Schoenberg Theorem to find a knot sequence for a basis of B-splines for a given vector space of splines
- Find the dimension of a vector space of splines given a knot sequence for a shifted power basis or a B-spline basis
- Write a B-spline as a sum of two lower degree B-splines using the recursion formula
- Write a B-spline as a sum of shifted power functions by using Cramer's rule and determinants
- Identify the lowest degree shifted power function coefficient in a B-spline as a nonzero confluent Vandermonde determinant
- Identify the index J used in the DeBoor algorithm to compute a B-spline curve at some value t
- Write a shifted power basis with increasing degree order
- Identify the nonzero values in the coordinate vector of a B-spline with respect to the shifted power basis

- Apply the Schoenberg-Whitney Theorem to determine if a set of points is correctly chosen to give a unique interpolant in a vector space of splines
- Write a basis of a total degree polynomial vector space in two variables
- Write a basis of a tensor product of function spaces in two variables
- Identify tensor products of polynomials inside the larger vector space of total degree polynomials