Lecture 27

Main Points:
- Introduction to Surfaces
- Review for Final

Total degree polynomial surfaces

Let \( V = P_d \) denote the vector space of polynomials in the variables \( x \) and \( y \) with total degree at most \( d \). A basis of \( V \) is:

\[
\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots, x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}.
\]

The dimension of \( V \) is:

\[
\text{dim}(V) = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}.
\]

Tensor Product Surfaces

The tensor product of two vector spaces of functions \( V \) and \( W \), with bases \( B_V = \{f_1, \ldots, f_n\} \) and \( B_W = \{g_1, \ldots, g_m\} \) respectively, is defined as the vector space \( V \otimes W \) with basis \( \{h_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \), where the basis functions \( h_{i,j}(x, y) = f_i(x)g_j(y) \). Any function \( h \) in \( V \otimes W \) can be expressed a sum:

\[
h = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j}.
\]

So, the definition of tensor product space is:

\[
V \otimes W = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j} : h_{i,j}(x, y) = f_i(x)g_j(y), f_i \in B_V, g_j \in B_W \right\}.
\]

The dimension of a tensor product space is:

\[
\text{dim}(V \otimes W) = m \cdot n = \text{dim}(V) \cdot \text{dim}(W).
\]

A tensor product surface is the graph of \( z = h(x, y) \) with \( h \in V \otimes W \).

The space \( P_d \otimes P_d \) is a subspace of \( P_{2d,\{x,y\}} \).

Similarly, the space \( P_m \otimes P_n \) is a subspace of \( P_{m+n,\{x,y\}} \).

Examples:
- Let \( V = P_2 \) with variable \( x \), and let \( W = P_3 \) with variable \( y \). Then the tensor product \( V \otimes W \) has basis:

\[
\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^3, x^2y^2, x y^3, x^2 y^3, x y^5, x^3 y^3, x^2 y^4, x y^5, y^6\}.
\]

This is a subset of the basis of \( P_5,\{x,y\} \):

\[
\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, x^3y^3, x^2y^3, x y^4, y^5\}.
\]
Points to be able to work out for the Final Exam:

- Find the polar form of a polynomial in Standard basis
- Find the polar form of a polynomial in Bernstein, or other, bases
- Identify the three defining properties of polar forms
- Verify if a given function satisfies one or more of the defining properties of polar forms
- Write the polar form for a parametric polynomial curve from the standard form
- Write the polar form for a Bezier curve from the BB-form
- Apply the control point property to find control points of a parametric polynomial curve in standard basis form
- Apply the reparametrization property to find control points of a reparametrized curve using the polar form of the original
- Use Nested Linear Interpolation to evaluate a polar form
- Verify equivalence of BB-form and NLI-form
- Write a Bezier curve in Cumulative form
- Find the derivative of a Bezier curve using Cumulative form
- Recognize a degenerate Bezier curve
- Classify the type of an implicit quadratic by using the Discriminant
- Use the five point construction to find a quadratic equation
- Use the tangent construction to find an implicit equation of a Bezier curve
- Use the tangent line properties of quadratic Bezier curves to solve for one control point given the other two
- Write the standard basis for a vector space of splines with continuity and multiplicity vectors
- Use a knot sequence to write a shifted power basis
- Work out a $B$-spline from the divided difference formula
- Write the knot subsequence for a $B$-spline based on a knot sequence and starting index
- Find the support of a $B$-spline (where it is nonzero)
- Find the exact order of continuity of a $B$-spline at one of its knot values using the multiplicity
- Use the Curry-Schoenberg Theorem to find a knot sequence for a basis of $B$-splines for a given vector space of splines
- Find the dimension of a vector space of splines given a knot sequence for a shifted power basis or a $B$-spline basis
- Write a $B$-spline as a sum of two lower degree $B$-splines using the recursion formula
- Write a $B$-spline as a sum of shifted power functions by using Cramer’s rule and determinants
- Identify the lowest degree shifted power function coefficient in a $B$-spline as a nonzero confluent Vandermonde determinant
- Identify the index $J$ used in the DeBoor algorithm to compute a $B$-spline curve at some value $t$
- Write a shifted power basis with increasing degree order
- Identify the nonzero values in the coordinate vector of a $B$-spline with respect to the shifted power basis
• Apply the Schoenberg-Whitney Theorem to determine if a set of points is correctly chosen to give a unique interpolant in a vector space of splines

• Write a basis of a total degree polynomial vector space in two variables

• Write a basis of a tensor product of function spaces in two variables

• Identify tensor products of polynomials inside the larger vector space of total degree polynomials