Lecture 29

Main Points:

- Derivatives of B-spline functions

B-splines of degree d for a knot sequence

Given a knot sequence $\mathbf{t}$ with $t_0 \le t_1 \le \ldots \le t_N$, we define for any degree $d \ge 0$, and for $0 \le i \le N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d$$

The set $\mathcal{B}_d(\mathbf{t})$ or $\mathcal{B}_{d,\mathbf{t}}$ is the set of B-splines associated to the knot sequence $\mathbf{t}$: $\mathcal{B}_d(\mathbf{t}) = \{\mathcal{B}_0^d(t), \ldots, \mathcal{B}_{N-d-1}^d(t)\}$

The B-spline recursion formula

The (DeBoor-Cox) recursion formula for B-splines of degree d associated to a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$ is:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i}\mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}}\mathcal{B}_{i+1}^{d-1}(t)$$

The base case is:

$$\mathcal{B}_i^0(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & elsewhere \end{cases}$$

Special case: if $t_{i+d+1} = t_i$ then the B-spline $\mathcal{B}_i^d(t)$ is defined to be zero.

Note: The base case and special case above are necessary if this is taken to be the definition of B-splines, which it is in many texts. However, the recursion follows directly from the definition using the divided differences. We have already worked out the base case $d = 0$ with the definition.

Derivatve of a B-spline function

$$\frac{d}{dt}\mathcal{B}_i^d(t) = d\left(\frac{\mathcal{B}_i^{d-1}(t)}{t_{i+d} - t_i} - \frac{\mathcal{B}_{i+1}^{d-1}(t)}{t_{i+d+1} - t_{i+1}}\right)$$

Proof:

First recall that we expressed the B-spline $\mathcal{B}_i^d(t)$ in terms of truncated power functions using the interpretation of the bracket operator as a single coefficient which can be obtained with Cramer's Rule. The truncated power functions are of the form

$$(t - t_j)_+^d,\ (t - t_j)_+^{d-1},\ \ldots$$

where t_j is a knot value in the sequence $t_i, \ldots, t_{i+d+1}$, and the lower degree truncated powers occur as derivatives of the first one. The number of derivatives used is determined by the multiplicity of knots, in the usual way.

So, it is certainly possible to differentiate the B-spline by first writing it as a sum of truncated powers, and differentiating term by term. However, we can also simply differentiate the function $(t-x)_+^d$ which occurs in the definition of the B-spline, treating x as a constant, and then apply the bracket operator to the result. This will have exactly the same effect as applying the bracket operator and then differentiating. This can be summarized by saying the differentiation commutes with the bracket operator. This is a fancy way of saying that because differentiation is a linear operator on functions, and so is the bracket operator, we can apply them in any order and achieve the same result.

$$
\begin{aligned}
\frac{d}{dt}\mathcal{B}_i^d(t) &= \frac{d}{dt}\left[(-1)^{d+1}(t_{i+d+1}-t_i)[t_i,t_{i+1},\ldots,t_{i+d+1}](t-x)_+^d\right] \\
&= (-1)^{d+1}(t_{i+d+1}-t_i)[t_i,t_{i+1},\ldots,t_{i+d+1}]d(t-x)_+^{d-1} \\
&= d(-1)^{d+1}(t_{i+d+1}-t_i)\left(\frac{[t_{i+1},\ldots,t_{i+d+1}](t-x)_+^{d-1}-[t_i,\ldots,t_{i+d}](t-x)_+^{d-1}}{t_{i+d+1}-t_i}\right) \\
&= d(-1)^{d+1}\left(\frac{\mathcal{B}_{i+1}^{d-1}(t)}{(-1)^d(t_{i+d+1}-t_{i+1})}-\frac{\mathcal{B}_i^{d-1}(t)}{(-1)^d(t_{i+d}-t_i)}\right) \\
&= d\left(\frac{\mathcal{B}_i^{d-1}(t)}{t_{i+d}-t_i}-\frac{\mathcal{B}_{i+1}^{d-1}(t)}{t_{i+d+1}-t_{i+1}}\right)
\end{aligned}
$$

This completes the proof. □

We can also write the derivative of a sum of B-splines as follows:

Let $\mathbf{t}$ be a knot sequence with B-splines $\mathcal{B}_i^d(t)$, $i=0,\ldots,N-d-1$. If f is equal a linear combination of these B-splines:

$$f(t)=\sum_{i=0}^{N-d-1}c_i\mathcal{B}_i^d(t)$$

then we have:

$$f'(t)=d\sum_{i=0}^{N-d}\frac{c_i-c_{i-1}}{t_{i+d}-t_i}\mathcal{B}_i^{d-1}(t),$$

where $c_{-1}=c_{N-d}=0$ are extra zero coefficients.