Lecture 5

Main Points:

- Ordered pairs of polynomials
- Piecewise polynomial (spline) vector spaces

The vector space P_d^2:

P_d^2 is simply the vector space of ordered pairs of polynomials of degree $\leq d$.

Addition and scalar multiplication in P_d^2 are defined componentwise:

$$(p_1(t), p_2(t)) + (q_1(t), q_2(t)) = (p_1(t) + q_1(t), p_2(t) + q_2(t)),$$

$$c(p_1(t), p_2(t)) = (cp_1(t), cp_2(t)).$$

Examples:

- some elements of P_2^2 are: $(2 - t + 3t^2, 1 - t)$ and $(3 + t^2, 5 - 2t - 7t^2)$.
- some elements of P_3^2 are: $(1 - 5t + 3t^3, 1 - 4t^2)$ and $(3 + 2t^2, 5 - 2t - 7t^2)$. Note: the latter ordered pair has no degree 3 terms, which is fine since the degrees must simply be at most 3.

A basis for P_2^2:

A basis for P_2^2 can be given by:

$$\{(1, 0), (t, 0), (t^2, 0), (0, 1), (0, t), (0, t^2)\}.$$

We can check that this set spans P_2^2, and is also linearly independent. The spanning property is easily seen since:

$$\begin{aligned}(p_1(t), p_2(t)) &= (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \\ &= a_0(1, 0) + a_1(t, 0) + a_2(t^2, 0) + b_0(0, 1) + b_1(0, t) + b_2(0, t^2).\end{aligned}$$

The linear independence can be seen by appealing to the definition of linear independence. We need to show that any linear combination of the ordered pairs can only be zero if all the coefficients are zero. Suppose we have such a linear combination set equal to the zero vector:

$$a_0(1, 0) + a_1(t, 0) + a_2(t^2, 0) + b_0(0, 1) + b_1(0, t) + b_2(0, t^2) = (0, 0).$$

This is equivalent to having:

$$(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) = (0, 0),$$

which in turn implies that

$$a_0 + a_1 t + a_2 t^2 = 0, \quad \text{and} \quad b_0 + b_1 t + b_2 t^2 = 0.$$

Since these are both linear combinations in P_2 of the standard basis polynomials, we know that all of the coefficients must be zero, which follows from the linear independence of 1, t, and t^2 in P_2.

Other bases for P_2^2:

We can mimic the previous basis which used the standard basis polynomials of P_2, and switch to the Bernstein basis of P_2, to get the basis of P_2^2:

$$\begin{aligned}&\{(B_0^2(t),0),(B_1^2(t),0),(B_2^2(t),0),(0,B_0^2(t)),(0,B_1^2(t)),(0,B_2^2(t))\}.\\ &=\{((1-t)^2,0),(2(1-t)t,0),(t^2,0),(0,(1-t)^2),(0,2(1-t)t),(0,t^2)\}.\end{aligned}$$

Similarly, this can be done with any other basis of polynomials, such as $\{b_0(t),b_1(t),b_2(t)\}$, to form the basis of P_2^2:

$$\{(b_0(t),0),(b_1(t),0),(b_2(t),0),(0,b_0(t)),(0,b_1(t)),(0,b_2(t))\}.$$

For example, we could take $\{b_0(t),b_1(t),b_2(t)\}$ equal to a Vandermonde basis of P_2 such as $\{(t-1)^2,(t-2)^2,(t-3)^2\}$ to get:

$$\{((t-1)^2,0),((t-2)^2,0),((t-3)^2,0),(0,(t-1)^2),(0,(t-2)^2),(0,(t-3)^2)\}.$$

Basis for P_d^k:

The set of k-tuples

$$(t^i,0,\dots,0),(0,t^i,0,\dots,0),\dots,(0,\dots,0,t^i)$$

for $i=0\dots,d$, is a basis of P_d^k. There are k entries in the list for each i, so $(d+1)k$ in all, which says that the dimension of P_d^k is $(d+1)k$.

Examples:

- P_1^3 has dimension 6 with basis:
$$\{(1,0,0),(0,1,0),(0,0,1),\\(t,0,0),(0,t,0),(0,0,t)\}$$
- P_3^4 has dimension 16 with basis:
$$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),\\(t,0,0,0),(0,t,0,0),(0,0,t,0),(0,0,0,t),\\(t^2,0,0,0),(0,t^2,0,0),(0,0,t^2,0),(0,0,0,t^2),\\(t^3,0,0,0),(0,t^3,0,0),(0,0,t^3,0),(0,0,0,t^3)\}$$
- P_2^5 has dimension 15 with basis:
$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),\\(t,0,0,0,0),(0,t,0,0,0),(0,0,t,0,0),(0,0,0,t,0),(0,0,0,0,t),\\(t^2,0,0,0,0),(0,t^2,0,0,0),(0,0,t^2,0,0),(0,0,0,t^2,0),(0,0,0,0,t^2)\}$$

Next, we develop a different basis of P_d^k. We start by including all the elements:

$$(1,1,\dots,1),(t,t,\dots,t),\dots,(t^d,t^d,\dots,t^d)$$

These are $d+1$ elements, corresponding to the standard basis of P_d. By choosing coefficients for these elements we can obtain any polynomial of degree d or less, repeated on each component. In order to form a basis we need to obtain any element of P_d^k of the form:

$$(p_1,p_2,\dots,p_k).$$

We can do this with the following additional elements:

$$(0,1,\dots,1),(0,t,\dots,t),\dots,(0,t^d,\dots,t^d),$$

$$
\begin{gathered}
(0,0,1,\ldots,1),(0,0,t,\ldots,t),\ldots,(0,0,t^d,\ldots,t^d),\\
\vdots\\
(0,\ldots,0,1),(0,\ldots,0,t),\ldots,(0,\ldots,0,t^d).
\end{gathered}
$$

In order to see that the above elements span P_d^k, we first write:

$$
\begin{aligned}
(p_1,p_2,\ldots,p_k) &= (p_1,p_1,\ldots,p_1)\\
&= +(0,p_2-p_1,\ldots,p_2-p_1)\\
&= +(0,0,p_3-p_2\ldots,p_3-p_2)\\
&= +(0,0,0,p_4-p_3\ldots,p_4-p_3)\\
&= \vdots\\
&= +(0,0,0,\ldots,0,p_k-p_{k-1})
\end{aligned}
$$

Note: In the above sum each line makes a correction to the previous line so that one more component has the desired polynomial. Also, each line can be spanned by exactly the corresponding line in the following basis:

$$
\begin{gathered}
\{(1,1,\ldots,1),(t,t,\ldots,t),\ldots,(t^d,t^d,\ldots,t^d),\\
(0,1,\ldots,1),(0,t,\ldots,t),\ldots,(0,t^d,\ldots,t^d),\\
(0,0,1,\ldots,1),(0,0,t,\ldots,t),\ldots,(0,0,t^d,\ldots,t^d),\\
\vdots\\
(0,\ldots,0,1),(0,\ldots,0,t),\ldots,(0,\ldots,0,t^d)\}
\end{gathered}
$$

Examples:

- P_1^3 has dimension 6 with basis:

$$\{(1,1,1),(t,t,t),(0,1,1),(0,t,t),(0,0,1),(0,0,t)\}$$

- Write the element $(2t-1,3,5t)$ in P_1^3 in terms of the two bases given so far. Call these two bases:

$$B_1=\{(1,0,0),(t,0,0),(0,1,0),(0,t,0),(0,0,1),(0,0,t)\}$$

  and

$$B_2=\{(1,1,1),(t,t,t),(0,1,1),(0,t,t),(0,0,1),(0,0,t)\}.$$

  In the first case:

$$(2t-1,3,5t)=-(1,0,0)+2(t,0,0)+3(0,1,0)+0(0,t,0)+0(0,0,1)+5(0,0,t).$$

  For the second case, we need to write

$$p_1(t)=2t-1,\ \ p_2(t)=3,\ \text{ and } p_3(t)=5t.$$

  Then we also have

$$p_2(t)-p_1(t)=4-2t,\ \text{ and }\ p_3(t)-p_2(t)=5t-3.$$

  Now we can follow the method above for

$$(p_1,p_2,p_3)=(p_1,p_1,p_1)+(0,p_2-p_1,p_2-p_1)+(0,0,p_3-p_2)$$

  to obtain:

$$(2t-1,3,5t)=-(1,1,1)+2(t,t,t)+4(0,1,1)-2(0,t,t)-3(0,0,1)+5(0,0,t).$$

- Find the coordinate vectors of the element $(2t-1, 3, 5t)$ in P_1^3 with respect to the bases B_1 and B_2. Based on the coefficients from the previous example we have the coordinate vectors:

$$B_1 : \begin{pmatrix} -1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 5 \end{pmatrix} \text{ and } B_2 : \begin{pmatrix} -1 \\ 2 \\ 4 \\ -2 \\ -3 \\ 5 \end{pmatrix}.$$

- P_3^4 has dimension 16 with basis:

$$\{(1,1,1,1), (t,t,t,t), (t^2,t^2,t^2,t^2), (t^3,t^3,t^3,t^3),$$
$$(0,1,1,1), (0,t,t,t), (0,t^2,t^2,t^2), (0,t^3,t^3,t^3),$$
$$(0,0,1,1), (0,0,t,t), (0,0,t^2,t^2), (0,0,t^3,t^3),$$
$$(0,0,0,1), (0,0,0,t), (0,0,0,t^2), (0,0,0,t^3)\}$$

- P_2^5 has dimension 15 with basis:

$$\{(1,1,1,1,1), (t,t,t,t,t), (t^2,t^2,t^2,t^2,t^2),$$
$$(0,1,1,1,1), (0,t,t,t,t), (0,t^2,t^2,t^2,t^2),$$
$$(0,0,1,1,1), (0,0,t,t,t), (0,0,t^2,t^2,t^2),$$
$$(0,0,0,1,1), (0,0,0,t,t), (0,0,0,t^2,t^2),$$
$$(0,0,0,0,1), (0,0,0,0,t), (0,0,0,0,t^2)\}$$

Piecewise polynomial functions:

A piecewise polynomial function (or ppf) is a function defined on a sequence of intervals by various polynomials. If the sequence of intervals is $[u_0, u_1], [u_1, u_2], \dots, [u_{k-1}, u_k]$, then we denote this sequence with the notation:

$$[u_0, u_1, \dots, u_k].$$

The endpoints of this sequence of intervals are u_0 and u_k, and all the other values u_i for $i = 1 \dots, k-1$ are called *breakpoints*. A ppf defined on this sequence of intervals has the following general form, where $p_1(t), \dots, p_k(t)$ are polynomials. Note that the intervals which are used by each polynomial are closed on the left and open on the right for p_1 through p_{k-1} and then closed on both ends for p_k. This is done so that the function f is defined exactly once for all points in the closed interval $[u_0, u_k]$.

$$f(t) = \begin{cases} p_1(t), & u_0 \le t < u_1 \\ p_2(t), & u_1 \le t < u_2 \\ \vdots & \vdots \\ p_k(t), & u_{k-1} \le t \le u_k \end{cases}$$

Examples:

- A ppf defined by cubic polynomials on the sequence of intervals $[0, 2, 4]$:

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t - 5, & 0 \leq t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \leq t \leq 4 \end{cases}$$

- A ppf defined by linear polynomials on the sequence of intervals $[0, 1, 2, 3, 4]$:

$$f(t) = \begin{cases} p_1(t) = 2t - 5, & 0 \leq t < 1 \\ p_2(t) = 4, & 1 \leq t < 2 \\ p_3(t) = t - 1, & 2 \leq t < 3 \\ p_4(t) = 3t + 7, & 3 \leq t \leq 4 \end{cases}$$

The piecewise polynomial vector space $P_d^k[u_0, u_1, \ldots, u_k]$.

The set of all piecewise polynomial functions on the sequence of intervals $[u_0, u_1, \ldots, u_k]$ forms the vector space which we call $P_d^k[u_0, u_1, \ldots, u_k]$. Addition and scalar multiplication in this vector space are given simply by addition of functions and real number multipication of functions. So this vector space is a subspace of the vector space $\mathcal{F}[u_0, u_k]$ of all functions on the interval $[u_0, u_k]$.

Correspondence between P_d^k and $P_d^k[u_0, u_1, \ldots, u_k]$:

Any ordered k-tuple of polynomials $(p_1(t), p_2(t), \ldots, p_k(t))$ in P_d^k can be converted into a ppf in $P_d^k[u_0, u_1, \ldots, u_k]$, given by:

$$f(t) = \begin{cases} p_1(t), & u_0 \leq t < u_1 \\ p_2(t), & u_1 \leq t < u_2 \\ \vdots & \vdots \\ p_k(t), & u_{k-1} \leq t \leq u_k \end{cases}$$

This process can also be reversed by taking this ppf and stripping away all of the information about intervals and forming the ordered k-tuple of the polynomials.

Examples:

- The element $(t, 1, 2 - t)$ in P_1^3 corresponds to the ppf

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 2 - t, & 2 \leq t \leq 3 \end{cases}$$

  in $P_1^3[0, 1, 2, 3]$.

- The element $(1 - 3t + t^2, 4, 2 - t^2, t + 5)$ in P_2^4 corresponds to the ppf

$$f(t) = \begin{cases} 1 - 3t_t^2, & 0 \leq t < 3 \\ 4, & 3 \leq t < 5 \\ 2 - t^2, & 5 \leq t < 6 \\ t + 5, & 6 \leq t \leq 8 \end{cases}$$

  in $P_1^3[0, 3, 5, 6, 8]$.

Isomorphism between P_d^k and $P_d^k[u_0, u_1, \ldots, u_k]$:

The above correspondence gives a dictionary between the ordered k-tuples of polynomials, and ppf's on a given sequence of intervals. It is important to note that this correspondence preserves linear algebra constructs. In particular, if

$$(p_1(t), p_2(t), \ldots, p_k(t)) \text{ and } (q_1(t), q_2(t), \ldots, q_k(t))$$

have corresponding ppf's $f(t)$ and $g(t)$, and

$$(p_1(t), p_2(t), \ldots, p_k(t)) + (q_1(t), q_2(t), \ldots, q_k(t)) = (r_1(t), r_2(t), \ldots, r_k(t))$$

and

$$f(t) + g(t) = h(t),$$

then it must the case that the ordered k-tuple

$$(r_1(t), r_2(t), \ldots, r_k(t))$$

corresponds to the ppf $h(t)$. Further, all linear algebra calculations can be transferred back and forth between these two vector spaces under this correspondence.

Modified bases for P_d^k:

Since we have many bases for P_d, it is also possible to use these to find other bases for P_d^k. In general, if $b_0(t), \ldots, b_d(t)$ is any basis of P_d, then we can construct the set of k-tuples

$$(b_i(t), 0, \ldots, 0), (0, b_i(t), 0, \ldots, 0), \ldots, (0, \ldots, 0, b_i(t))$$

for $i = 0 \ldots, d$, which is a basis of P_d^k. There are k entries in the list for each i, so $(d+1)k$ in all, which again exhibits the fact that the dimension of P_d^k is $(d+1)k$.