Lecture 7

Main Points:

- The General Interpolation Problem
- Polynomial Interpolation
- Existence and Uniqueness of interpolating polynomial in P_d

The General Interpolation Problem:

The general interpolation problem assumes that we are working with a vector space of functions V, with finite dimension n. Then we suppose that we are given n "data points": $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, which are interpreted as the desired input and output for some function in V. In other words, the problem is to find a function f in V such that:

$$f(x_1) = y_1, f(x_2) = y_2, \ldots, f(x_n) = y_n.$$

If it exists, such a function is often called an *interpolant* to the data that we specified. If we also have a basis B for V, say:

$$B = \{b_1(t), \ldots, b_n(t)\}$$

then we can phrase the interpolation problem: to find coefficients $a_1, \ldots, a_n$ so that the function

$$f(t) = a_1 b_1(t) + \cdots + a_n b_n(t)$$

satisfies the conditions:

$$f(x_1) = y_1, f(x_2) = y_2, \ldots, f(x_n) = y_n.$$

This problem can in turn be converted to a linear system $Ax = b$ where the coefficients $a_1, \ldots, a_n$ are the unkowns. There will be a solution, for the coefficients and hence for f, if and only if the matrix A is invertible. Such a function may or may not exist depending on the vector space V and the distribution of the data points.

The Polynomial Interpolation Problem:

For polynomial interpolation, we use $V = P_d$ and the variable t. We also may specify the data points by choosing distinct input values $t_0, \ldots, t_d$ and then giving a data function $g(t)$ for the outputs. This is often done because it is natural to use a polynomial $p(t)$ to approximate some other function $g(t)$. The polynomial interpolation problem then takes the form:

Given (distinct) values $t_0, t_1, \ldots, t_d$, and a data function $g(t)$, find $p(t) \in P_d$ satisfying: $p(t_i) = g(t_i)$ for $i = 0, \ldots, d$.

Existence and Uniqueness of the Interpolating Polynomial:

It is a fact that given any (distinct) values $t_0, t_1, \ldots, t_d$, and a data function $g(t)$, there always exists exactly one polynomial $p(t) \in P_d$ satisfying: $p(t_i) = g(t_i)$ for $i = 0, \ldots, d$.

Examples:

- Let $t_0 = 1$ and $t_1 = 3$. Also let $g(t) = 3/t$, so that $g(1) = 3$ and $g(3) = 1$. Then the interpolation problem asks for a polynomial $p(t)$ of degree $d = 1$ that satisfies $p(1) = g(1)$ and $p(3) = g(3)$, or in other words the line through the two points $(1, 3)$ and $(3, 1)$, which is

  $$p(t) = -t + 4.$$

  Since this clearly is the only line through the two points, this answer is unique.

Proofs of Existence and Uniqueness of the Interpolating Polynomial:

We will prove the existence and uniqueness in two ways:

1. with the standard basis, and a linear system with Vandermonde determinant,
2. with the Lagrange basis.

1. Standard Basis Proof

When we first introduced the Vandermonde determinant, we did not say where it most naturally arises. The answer is that it comes up naturally in the context of solving a linear system in order to find an interpolating polynomial with respect to the standard basis.

Given (distinct) values $t_0, t_1, \ldots, t_d$, and a data function $g(t)$, we seek a polynomial

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d$$

which has the property that $p(t_i) = g(t_i)$ for $i = 0, \ldots, d$. In order to solve for the coefficients of such a $p(t)$, we set up a linear system with one row for each i of the form:

$$a_0 + a_1 t_i + \cdots + a_d t_i^d = g(t_i).$$

Since the a_j's are the variables, and t_i is a constant, we can also write this as

$$1 \cdot a_0 + t_i \cdot a_1 + \cdots + t_i^d \cdot a_d = g(t_i).$$

Extracting the constants as coefficients of the a_j, we can convert this to a row of an augmented matrix:

$$[1 \ \ t_i \ \ t_i^2 \ \ \cdots \ \ t_i^d \ \mid \ g(t_i)].$$

The entire augmented matrix looks like this:

$$\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d & | & g(t_0) \\ 1 & t_1 & t_1^2 & \cdots & t_1^d & | & g(t_1) \\ 1 & t_2 & t_2^2 & \cdots & t_2^d & | & g(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & | & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d & | & g(t_d) \end{pmatrix}.$$

As a matrix equation, it is equivalent to:

$$\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} g(t_0) \\ g(t_1) \\ g(t_2) \\ \vdots \\ g(t_d) \end{pmatrix}.$$

We know that this matrix has Vandermonde determinant equal to the product

$$\prod_{0 \le i < j \le d} (t_j - t_i),$$

which is clearly not equal to zero as long as the t_i are all distinct, which is required since we ask for $d+1$ distinct inputs in the interpolation problem.

Since the determinant is nonzero, the matrix is invertible and the linear system has exactly one solution. So there is exactly one solution for the coefficients of the polynomial $p(t)$. This completes the first proof of existence and uniqueness for the interpolating polynomial.

2. Lagrange Basis Proof

First we define the Lagrange basis of P_d. Given a sequence of distinct numbers $t_0, t_1, \ldots, t_d$ let

$$L_i(t) = \frac{(t-t_0)(t-t_1)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_d)}{(t_i-t_0)(t_i-t_1)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_d)}, \quad i = 0, \ldots, d.$$

For example, if $d = 2$ then

$$L_0(t) = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}, \quad L_1(t) = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}, \quad L_2(t) = \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)}.$$

An immediate property of the Lagrange polynomials is:

$$L_i(t_i) = 1, \ i = 0, \ldots, d \quad \text{and} \quad L_i(t_j) = 0, \ i \neq j.$$

It is a fact that the set of Lagrange polynomials is a basis of P_d:

$$\{L_0(t), L_1(t), \ldots, L_d(t)\}.$$

We can prove this fact by showing that they are linearly independent. We appeal to the definition of linear independence. So, suppose that there is a linear combination of these polynomials equal to the zero polynomial (we write $0(t)$ for the zero polynomial to emphasize that it is not simply the number zero):

$$c_0 L_0(t) + c_1 L_1(t) + \cdots + c_d L_d(t) = 0(t).$$

To see that all the coefficients must be zero (which is required in the definition of linear independence) we simply plug in the numbers $t_0, t_1, \ldots, t_d$ to the equation, and use the above property, giving the equations: $c_0 = 0, c_1 = 0, \ldots, c_d = 0$ respectively.

Next, we write the interpolating polynomial to the data values $t_0, \ldots, t_d$ with data function $g(t)$ as:

$$p(t) = g(t_0)L_0(t) + g(t_1)L_1(t) + \cdots + g(t_p)L_p(t).$$

Again, we can simply use the above property, and plug in the data values, to see that indeed

$$p(t_i) = g(t_i), \ i = 0, \ldots, d.$$

This shows the existence of the interpolating polynomial in P_d. In order to show the uniqueness of $p(t)$, we suppose that there is another polynomial $q(t)$ in P_d satisfying:

$$q(t_i) = g(t_i), \ i = 0, \ldots, d.$$

Then we can write such a polynomial $q(t)$ in terms of the Lagrange basis as:

$$q(t) = b_0 L_0(t) + b_1 L_1(t) + \cdots + b_p L_p(t).$$

But then we also plug in the data values to this equation and find that

$$q(t_i) = b_i, \ i = 0, \ldots, d,$$

which implies in turn that

$$b_i = q(t_i) = g(t_i), \ i = 0, \ldots, d,$$

which means that

$$q(t) = g(t_0)L_0(t) + g(t_1)L_1(t) + \cdots + g(t_p)L_p(t) = p(t).$$

This shows that $p(t)$, the interpolating polynomial, is unique in P_d, and this completes the proof of the existence and uniqueness of the interpolating polynomial using the Lagrange basis.

Examples:

- Find the interpolating polynomial $p(t)$ in P_2 which matches the data function $g(t) = 2^t$ for data values $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$. With the standard basis we assume the form

$$p(t) = a_0 + a_1 t + a_2 t^2$$

  and we then get the linear system:

$$\begin{aligned} 1 \cdot a_0 + 0 \cdot a_1 + 0^2 \cdot a_2 &= g(0) \\ 1 \cdot a_0 + 1 \cdot a_1 + 1^2 \cdot a_2 &= g(1) \\ 1 \cdot a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 &= g(2) \end{aligned}$$

  which also simplifies to the matrix equation:

$$\begin{pmatrix} 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

  The augmented matrix then reduces with Gaussian elimination to:

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 1 & 1 & 1 & | & 2 \\ 1 & 2 & 4 & | & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 2 & 4 & | & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 2 & | & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1/2 \\ 0 & 0 & 1 & | & 1/2 \end{pmatrix}$$

  which tells us that the interpolating polynomial is:

$$p(t) = 1 + \frac{1}{2}t + \frac{1}{2}t^2.$$

- Now we find $p(t)$ with the same data using the Lagrange polynomials:

$$\begin{aligned} p(t) &= g(0)L_0(t) + g(1)L_1(t) + g(2)L_2(t) \\ &= 1 \cdot \frac{(t-1)(t-2)}{(0-1)(0-2)} + 2 \cdot \frac{(t-0)(t-2)}{(1-0)(1-2)} + 4 \cdot \frac{(t-0)(t-1)}{(2-0)(2-1)} \\ &= \frac{1}{2}(t-1)(t-2) - 2t(t-2) + 2t(t-1) \\ &= \frac{1}{2}(t^2 - 3t + 2) - 2t^2 + 4t + 2t^2 - 2t \\ &= 1 + \frac{1}{2}t + \frac{1}{2}t^2. \end{aligned} \tag{1}$$