Lecture 9

Main Points:

- Proof of Newton form
- Proof of Leibniz' Rule
- Osculating polynomials

Proof of the Newton form:

To establish the validity of the Newton form, we suppose that the interpolating polynomial $p(t)$ with (distinct) data values $t_0, \ldots, t_d$ and data function $g(t)$ exists and is unique in P_d, and also that the interpolating polynomial $p_0(t)$ with the subset of data values $t_0, t_1, \ldots, t_{d-1}$ exists and is unique in P_{d-1}. For convenience, we will rename the polynomial $p_0(t)$ as $q_{d-1}(t)$, to indicate that it has degree $d-1$. Then we can consider the polynomial

$$f(t) = p(t) - q_{d-1}(t),$$

which is also in P_d, and has the property

$$f(t_i) = p(t_i) - q_{d-1}(t_i) = g(t_i) - g(t_i) = 0, \quad i = 0, \ldots, d-1.$$

This says that we can factor the polynomial $f(t)$ to obtain:

$$f(t) = C \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where C is the coefficient of t^d in this polynomial. But since $q_{d-1}(t)$ has degree at most $d-1$, we see that C is the coefficient of t^d in $p(t)$, which is by definition the divided difference:

$$C = [t_0, t_1, \ldots, t_d]g.$$

Thus we obtain:

$$p(t) - q_{d-1}(t) = [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

or:

$$p(t) = q_{d-1}(t) + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Now, the argument we just applied to $p(t)$ can also be carried out for $q_{d-1}(t)$, supposing that $q_{d-2}(t)$ is the interpolating polynomial with the data values: $t_0, t_1, \ldots, t_{d-2}$, and we obtain:

$$q_{d-1}(t) = q_{d-2}(t) + [t_0, t_1, \ldots, t_{d-1}]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-2}),$$

Continuing in this we way, we will eventually arrive at the statement

$$q_1(t) = q_0(t) + [t_0, t_1]g \cdot (t - t_0),$$

where $q_0(t)$ is the interpolating polynomial of degree 0 with the data value t_0 and data function $g(t)$, in other words $q_0(t) = g(t_0) = [t_0]g$ is constant. Piecing all of this back together, we arrive at the Newton form:

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Proof of Leibniz' Rule for Divided Differences

Let $f(t) = g(t)h(t)$. Then the general form of Leibniz' Rule looks like:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]f = \sum_{r=i}^{i+k}([t_i, \ldots, t_r]g)([t_r, \ldots, t_{i+k}]h).$$

For degree $d = 2$ the Leibniz formula looks like this:

$$[t_0, t_1, t_2]f = [t_0]g[t_0, t_1, t_2]h + [t_0, t_1]g[t_1, t_2]h + [t_0, t_1, t_2]g[t_2]h.$$

First let's check the base case, $d = 0$. In this case there is only one entry in each of the tables for f, g, and h. In particular, we have

$$[t_0]f = f(t_0) = g(t_0)h(t_0) = [t_0]g[t_0]h.$$

We can also do the linear case $d = 1$ which says that:

$$[t_0, t_1]f = [t_0]g \cdot [t_0, t_1]h + [t_0, t_1]g \cdot [t_1]h.$$

We can check this easily since the divided differences in this case are just the slope of a line through two points. In other words:

$$[t_0, t_1]f = \frac{f(t_1) - f(t_0)}{t_1 - t_0}, \qquad [t_0, t_1]g = \frac{g(t_1) - g(t_0)}{t_1 - t_0}, \qquad \text{and} \qquad [t_0, t_1]h = \frac{h(t_1) - h(t_0)}{t_1 - t_0}.$$

Then the right hand side of Leibniz' rule becomes:

$$\begin{aligned}
[t_0]g \cdot [t_0, t_1]h + [t_0, t_1]g \cdot [t_1]h &= g(t_0) \cdot \frac{h(t_1) - h(t_0)}{t_1 - t_0} + \frac{g(t_1) - g(t_0)}{t_1 - t_0} h(t_1) \\
&= \\
&= \frac{g(t_0)h(t_1) - g(t_0)h(t_0) + g(t_1)h(t_1) - g(t_0)h(t_1)}{t_1 - t_0} \\
&= \\
&= \frac{g(t_1)h(t_1) - g(t_0)h(t_0)}{t_1 - t_0} \\
&= \\
&= \frac{f(t_1) - f(t_0)}{t_1 - t_0} \\
&= \\
&= [t_0, t_1]f
\end{aligned}$$

For the degree $d = 2$ case, we give a proof which also extends to the higher degree cases, using the Newton forms with data values t_0, t_1, and t_2, and each of the data functions f, g, and h. Call these Newton forms $p(t)$, $q(t)$ and $r(t)$, respectively. We will write the Newton forms slightly differently, taking advantage of the fact that the order of the data values does not matter. In particular, we will reverse the order of the data values for the Newton form $r(t)$, with data function h. The divided difference tables for $q(t)$ and $r(t)$ then look like this:

$$\begin{array}{llll}
t_0 & [t_0]g & & \\
 & & [t_0, t_1]g & \\
t_1 & [t_1]g & & [t_0, t_1, t_2]g \\
 & & [t_1, t_2]g & \\
t_2 & [t_2]g & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{llll}
t_2 & [t_2]h & & \\
 & & [t_1, t_2]h & \\
t_1 & [t_1]h & & [t_0, t_1, t_2]h \\
 & & [t_0, t_1]h & \\
t_0 & [t_0]h & &
\end{array}$$

It is tempting to think that the product of the Newton forms q and r is equal to the Newton form p. This is because of the property:

$$q(t_i) \cdot r(t_i) = g(t_i) \cdot h(t_i) = f(t_i), \quad i = 0, 1, 2.$$

This says that the product $q(t)r(t)$ satisfies the interpolation conditions. However, this product is only guaranteed to have degree ≤ 4, but we know that $p(t)$ exists in P_2. But even though the product does not give the Newton form p, we can still obtain p by subtracting off part of the product.

From the above tables we obtain the two Newton forms:

$$q(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1),$$

and

$$r(t) = [t_2]h + [t_1, t_2]h \cdot (t - t_2) + [t_0, t_1, t_2]h \cdot (t - t_1)(t - t_2).$$

Then the product of these is:

$$\begin{aligned} q(t) \cdot r(t) &= ([t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1)) \\ & \quad \cdot ([t_2]h + [t_1, t_2]h \cdot (t - t_2) + [t_0, t_1, t_2]h \cdot (t - t_2)(t - t_1)) \\ &= F(t) + G(t) \end{aligned}$$

where

$$\begin{aligned} F(t) &= [t_0]g \cdot [t_2]h \quad \text{(degree 0 term)} \\ &+ [t_0]g \cdot [t_1, t_2]h \cdot (t - t_2) + [t_2]h \cdot [t_0, t_1]g \cdot (t - t_0) \quad \text{(degree 1 terms)} \\ &+ [t_0]g \cdot [t_0, t_1, t_2]h \cdot (t - t_2)(t - t_1) \\ &+ [t_0, t_1]g \cdot [t_1, t_2]h \cdot (t - t_0)(t - t_2) \quad \text{(degree 2 terms)} \\ &+ [t_0, t_1, t_2]g \cdot [t_2]h \cdot (t - t_0)(t - t_1) \end{aligned}$$

and

$$\begin{aligned} G(t) &= [t_0, t_1]g \cdot [t_0, t_1, t_2]h \cdot (t - t_0)(t - t_1)(t - t_2) \\ &+ [t_1, t_2]h \cdot [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1)(t - t_2) \quad \text{(degree 3 terms)} \\ &+ [t_0, t_1, t_2]g \cdot [t_0, t_1, t_2]h \cdot (t - t_0)(t - t_1)^2(t - t_2). \quad \text{(degree 4 term)} \end{aligned}$$

Next, we note that $G(t_0) = G(t_1) = G(t_2) = 0$. But then we have

$$\begin{aligned} q(t_i) \cdot r(t_i) &= g(t_i) \cdot h(t_i) = f(t_i) \\ &= F(t_i) + G(t_i) \\ &= F(t_i) \end{aligned}$$

for $i = 0, 1, 2$. In summary,

$$F(t_i) = f(t_i), \quad i = 0, 1, 2,$$

and $F(t)$ is in P_2. By the uniqueness of the interpolating polynomial we must have

$$F(t) = p(t).$$

Finally, to get the divided difference $[t_0, t_1, t_2]f$ we simply extract the coefficient of t^2 in $p(t)$, which is the same as $F(t)$. From the above we can see that the t^2 terms for $F(t)$ have exactly the coefficients predicted by Leibniz' rule:

$$[t_0, t_1, t_2]f = [t_0]g \cdot [t_0, t_1, t_2]h + [t_0, t_1]g \cdot [t_1, t_2]h + [t_0, t_1, t_2]g \cdot [t_2]h.$$

This completes the proof of Leibniz' rule for $d = 2$.

For the general case, we can use the same proof as for $d = 2$, obtaining the sum $F(t) + G(t)$, where $F(t)$ has degree $\leq d$ and $G(t_i) = 0$ for $i = 0, \ldots, d$. We can then extract Leibniz' rule from the sum of terms which contain t^d.

Examples:

- Let $f(t) = |t|(t-2)^2$, with $g(t) = |t|$, and $h(t) = (t-2)^2$, and take $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$. We then have the tables:

$$\begin{array}{cccc} -1 & 1 & & \\ & & -1 & \\ 0 & 0 & & 1 \\ & & 1 & \\ 1 & 1 & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} -1 & 9 & & \\ & & -5 & \\ 0 & 4 & & 1 \\ & & -3 & \\ 1 & 1 & & \end{array}$$

  Then we have the corresponding Leibniz formula for $[-1, 0, 1]f$:

$$\begin{aligned} [-1,0,1]f &= [-1]g[-1,0,1]h + [-1,0]g[0,1]h + [-1,0,1]g[1]h \\ &= (1)(1) + (-1)(-3) + (1)(1) \\ &= 5 \end{aligned}$$

  We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc} -1 & 9 & & \\ & & -9 & \\ 0 & 0 & & 5 \\ & & 1 & \\ 1 & 1 & & \end{array}$$

  This confirms directly that $[-1, 0, 1]f = 5$.

- Let $f(t) = (t-2)_+^2(t-2)$, with $g(t) = (t-2)_+^2$, and $h(t) = t-2$, and take $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$. We then have the tables:

$$\begin{array}{cccc} 1 & 0 & & \\ & & 0 & \\ 2 & 0 & & \frac{1}{2} \\ & & 1 & \\ 3 & 1 & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} 1 & -1 & & \\ & & 1 & \\ 2 & 0 & & 0 \\ & & 1 & \\ 3 & 1 & & \end{array}$$

  Then we have the corresponding Leibniz formula for $[1, 2, 3]f$:

$$\begin{aligned} [1,2,3]f &= [1]g[1,2,3]h + [1,2]g[2,3]h + [1,2,3]g[3]h \\ &= (0)(0) + (0)(1) + (\frac{1}{2})(1) \\ &= \frac{1}{2} \end{aligned}$$

  We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc}
1 & 0 & & \\
 & & 0 & \\
2 & 0 & & \frac{1}{2} \\
 & & 1 & \\
3 & 1 & &
\end{array}$$

Definition of the osculating polynomial

Instead of matching only values of a data function, we might want to also match derivative values. In the following definition, we take repeated data values to mean that we are requiring consecutive matching of derivatives. It turns out to be best to require derivatives in sequence, without any gaps, which is also referred to as *Hermite* interpolation.

Given any *nondecreasing* sequence of real numbers $t_0 \le t_1 \le \cdots \le t_d$ and a function $g(t)$ with values $g(t_i)$ at these numbers, suppose further that g is differentiable to order r_i at each t_i, where r_i is determined by $r_i = 0$ if $t_i < t_{i+1}$, and $r_i = k$ if $t_i = t_{i+1} = \cdots = t_{i+k}$ and $t_{i+k} < t_{i+k+1}$. Then define an *osculating polynomial* $p(t)$ with the data sequence $t_0, t_1, \cdots, t_d$ and data function $g(t)$ as a polynomial which satisfies:

$$p^{(j)}(t_i) = g^{(j)}(t_i) \;\; for \;\; i = 0, \ldots, d, \;\; and \;\; j = 0, \ldots, r_i.$$

Note: If we change the order of the sequence in such a way that equal values are still consecutive, the definition of the osculating polynomial is not affected. So we can allow changes in the order of the data as long as whenever $t_i = t_j$, with $i < j$, then also $t_i = t_k$ for all k satisfying $i < k < j$.

Examples:

- Find a polynomial which matches the data function $g(t) = \dfrac{1}{t-2}$ for the data sequence $t_0 = 0$, $t_1 = 0$, and $t_2 = 1$. This means that we want $p(t)$ in P_2 satisfying: $p(0) = g(0)$, $p'(0) = g'(0)$ and $p(1) = g(1)$. Since $g'(t) = \dfrac{-1}{(t-2)^2}$, we need to find $p(t)$ satisfying:

$$p(0) = -\frac{1}{2}, \;\; p'(0) = -\frac{1}{4} \qquad \text{and} \qquad p(1) = -1.$$

We can find such a $p(t)$ with the standard basis and a linear system: We solve for the coefficients a_0, a_1, and a_2 with

$$p(t) = a_0 + a_1 t + a_2 t^2, \qquad \text{and} \qquad p'(t) = a_1 + 2a_2 t.$$

The linear system is then:

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{4} \\ 1 & 1 & 1 & | & -1 \end{pmatrix}$$

which has reduced form:

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{4} \\ 0 & 0 & 1 & | & -\frac{1}{4} \end{pmatrix}$$

and solution $a_2 = \frac{1}{4}$, $a_1 + 2a_2 = -\frac{1}{4}$, or $a_1 = -\frac{3}{4}$, and $a_0 = -\frac{1}{2}$. So the osculating polynomial $p(t)$ is

$$p(t) = -\frac{1}{2} - \frac{1}{4}t - \frac{1}{4}t^2,$$

with

$$p'(t) = -\frac{1}{4} - \frac{1}{2}t.$$

Existence and Uniqueness of the osculating polynomial

It is a fact that for any nondecreasing sequence $t_0 \le t_1 \le \cdots \le t_d$ of real numbers, and function g, the osculating polynomial $p(t)$ with data function g and data values $t_0, t_1, \cdots, t_d$ exists as an element of P_d, and is unique.