MAT 320 Supplementary Notes Fall 2016

Discrete Fourier Transform

First we give the Discrete Fourier Transform, or DFT.

Setup: Let $\mathbf{x}$ be a signal of length N, which we think of as a complex vector in $\mathbb{C}^N$. This means that $\mathbf{x}$ has components $x_0, \ldots, x_{N-1}$ in $\mathbb{C}$ and we can write it as a complex column vector:

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}.$$

The DFT then gives a basis representation of $\mathbf{x}$ in terms of the Fourier basis, or DFT basis:

$$\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}\}$$

where

$$\mathbf{u}_k = \begin{pmatrix} e^{i\frac{2\pi}{N}k\cdot 0} \\ e^{i\frac{2\pi}{N}k\cdot 1} \\ e^{i\frac{2\pi}{N}k\cdot 2} \\ \vdots \\ e^{i\frac{2\pi}{N}k\cdot (N-1)} \end{pmatrix}.$$

Then we have the DFT as a dot product and a sum:

$$\mathrm{DFT}(\mathbf{x}, N, k) = \mathbf{x} \bullet \mathbf{u}_k = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \bullet \begin{pmatrix} e^{i\frac{2\pi}{N}k\cdot 0} \\ e^{i\frac{2\pi}{N}k\cdot 1} \\ e^{i\frac{2\pi}{N}k\cdot 2} \\ \vdots \\ e^{i\frac{2\pi}{N}k\cdot (N-1)} \end{pmatrix} = \sum_{t=0}^{N-1} x_t e^{-i\frac{2\pi}{N}kt}.$$

Note: Since the complex dot product uses a conjugate in each coordinate of the second vector, we see the minus sign appearing in the exponent in the sum, reflecting the fact that: $\overline{e^{i\theta}} = e^{-i\theta}$.

We can also use input/output signal notation. Given input signal $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$, then the Fourier Transform output signal $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$ has each coordinate given by:

$$X_k = \mathrm{DFT}(\mathbf{x}, N, k) = \sum_{t=0}^{N-1} x_t e^{-i\frac{2\pi}{N}kt}, \quad k = 0, \ldots, N-1.$$

It is also possible to reconstruct the signal components $x_0, x_1, \ldots, x_{N-1}$ by doing the inverse Fourier Transform:

$$x_t = \mathrm{DFT}^{-1}(\mathbf{X}, N, k) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i\frac{2\pi}{N}tk}, \quad t = 0, \ldots, N-1.$$

Recursive (Fast) Fourier Transform

Now assume $N = 2^m$ is a power of 2. Then we can compute each coordinate of $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$ in two pieces recursively, with each piece based on the half-size Fourier transforms as follows:

$$X_k = \mathrm{DFT}(\mathbf{x}_{ev}, N/2, k) + W_N^k \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k), \qquad k = 0, \ldots, \frac{N}{2} - 1$$

and

$$X_{\frac{N}{2}+k} = \mathrm{DFT}(\mathbf{x}_{ev}, N/2, k) - W_N^k \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k), \qquad k = 0, \ldots, \frac{N}{2} - 1,$$

where $\mathbf{x}_{ev} = (x_0, x_2, \ldots, x_{N-2})$ is the $N/2$ length signal of even index terms from $\mathbf{x}$, and $\mathbf{x}_{od} = (x_1, x_3, \ldots, x_{N-1})$ is the $N/2$ length signal of odd index terms from $\mathbf{x}$, and

$$W_N = e^{-i\frac{2\pi}{N}}.$$

Note: The ending case for the recursion is that the DFT of a length one signal is just the identity operator:

$$\mathrm{DFT}(x, 1, k) = x$$

for any x and any k.

This can be shown as in the textbook in the following steps:

First write:

$$X_k = \sum_{t=0}^{N-1} x_t e^{-i\frac{2\pi}{N}kt} = \sum_{t=0}^{\frac{N}{2}-1} x_{2t} e^{-i\frac{2\pi}{N}k(2t)} + \sum_{t=0}^{\frac{N}{2}-1} x_{2t+1} e^{-i\frac{2\pi}{N}k(2t+1)} = E_k + O_k$$

where we have defined E_k as the even-indexed terms and O_k as the odd-indexed terms.

Next, we write E_k and O_k as DFT's, using $M = \frac{N}{2}$:

$$E_k = \sum_{t=0}^{\frac{N}{2}-1} x_{2t} e^{-i\frac{2\pi}{N}k(2t)} = \sum_{t=0}^{M-1} x_{2t} e^{-i\frac{2\pi}{M}kt} = \mathrm{DFT}(\mathbf{x}_{ev}, N/2, k),$$

and

$$O_k = \sum_{t=0}^{\frac{N}{2}-1} x_{2t+1} e^{-i\frac{2\pi}{N}k(2t+1)} = e^{-i\frac{2\pi}{N}k} \cdot \sum_{t=0}^{M-1} x_{2t} e^{-i\frac{2\pi}{M}kt} = e^{-i\frac{2\pi}{N}k} \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k) = W_N^k \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k).$$

Next, we note that the functions

$$\mathrm{DFT}(\mathbf{x}_{ev}, N/2, k) \text{ and } \mathrm{DFT}(\mathbf{x}_{od}, N/2, k)$$

can only be called for the k values:

$$0 \le k \le \frac{N}{2} - 1.$$

So we can compute half of the X_k values as:

$$X_k = E_k + O_k = \mathrm{DFT}(\mathbf{x}_{ev}, N/2, k) + W_N^k \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k), \quad 0 \le k \le \frac{N}{2} - 1.$$

To get the other half, for $\frac{N}{2} \le k \le N-1$, we use the fact that the W_N^k values repeat:

$$W_N^{\frac{N}{2}+k} = W_N^{\frac{N}{2}} W_N^k = e^{-i\frac{2\pi}{N}(\frac{N}{2})} \cdot W_N^k = e^{-i\pi} W_N^k = -W_N^k.$$

So we get $W_N^{\frac{N}{2}+k} = -W_N^k$, and so:

$$X_{\frac{N}{2}+k} = \mathrm{DFT}(\mathbf{x}_{ev}, N/2, k) - W_N^k \cdot \mathrm{DFT}(\mathbf{x}_{od}, N/2, k), \quad 0 \le k \le \frac{N}{2} - 1.$$

Finally, also note that when calling DFT(_ , $N/2, k$) it happens that often $k > N/2$. This is not a problem since the basis vectors just repeat for indices $k > N/2$. So the DFT also just repeats, or in other words is equal to a DFT for a lower k value in $0 \le k \le N-1$. You can use a mod function if you want to, which always reduces k mod N. If you don't reduce mod N, it is taken care of by the powers of W since:

$$W_N^{k+N} = W_N^k W_N^N = W_N^k \cdot 1 = W_N^k.$$

This completes the recursive computation of each component of $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$.