Reson Filter Derivation of Peak Frequency Angle ω_m (Chapter 5.5)

Let $f(\omega)$ be the reciprocal of the magnitude response squared, written as a function of the angular frequency ω. So $|H(\omega)|^2 = 1/f$. We want to find the value $\omega = \omega_m$ where f achieves its minimum value on the interval $[0, \pi]$, which would correspond to the place where $1/f$ achieves its maximum. We will do this by simplifying the formula first, and then taking the first derivative to find critical points. After this, we will also take the second derivative and evaluate it at the critical point in the middle of the interval which is close to θ (the pole angle in the interval $[0, \pi]$). We will see that this is negative, confirming that f is concave up at this point, which then must be a minimum.

Then

$$
\begin{aligned}
f(\omega) &= \frac{1}{|H(\omega)|^2} = |(e^{i\omega} - Re^{i\theta})(e^{i\omega} - Re^{-i\theta})|^2 \\
&= |e^{i2\omega} - Re^{i(\omega+\theta)} - Re^{i(\omega-\theta)} + R^2|^2 \\
&= |(\cos 2\omega + i\sin 2\omega) - R(\cos(\omega+\theta) + i\sin(\omega+\theta)) - R(\cos(\omega-\theta) + i\sin(\omega-\theta)) + R^2|^2 \\
&= |(\cos 2\omega - R[\cos(\omega+\theta) + \cos(\omega-\theta)] + R^2) + i(\sin 2\omega - R[\sin(\omega+\theta) + \sin(\omega-\theta)])|^2 \\
&= |(\cos 2\omega - 2R\cos\omega\cos\theta + R^2) + i(\sin 2\omega - 2R\sin\omega\cos\theta)|^2 \\
&= (\cos 2\omega - 2R\cos\omega\cos\theta + R^2)^2 + (\sin 2\omega - 2R\sin\omega\cos\theta)^2 \\
&= \cos^2 2\omega + R^4 + 4R^2\cos^2\theta\cos^2\omega + 2R^2\cos 2\omega - 4R\cos\theta\cos\omega\cos 2\omega - 4R^3\cos\theta\cos\omega \\
&\quad + \sin^2 2\omega + 4R^2\cos^2\theta\sin^2\omega - 4R\cos\theta\sin\omega\sin 2\omega \\
&= 1 + R^4 + 4R^2\cos^2\theta + 2R^2(2\cos^2\omega - 1) - 4R\cos\theta\cos\omega(1 - 2\sin^2\omega) - 4R^3\cos\theta\cos\omega \\
&\quad -8R\cos\theta\sin^2\omega\cos\omega \\
&= 1 + R^4 + 4R^2\cos^2\theta - 2R^2 + 4R^2\cos^2\omega - 4R\cos\theta\cos\omega - 4R^3\cos\theta\cos\omega \\
&= (1 - R^2)^2 + 4R^2\cos^2\theta - 4R(1 + R^2)\cos\theta\cos\omega + 4R^2\cos^2\omega
\end{aligned}
$$

Next, we differentiate in order to find critical points for f:

$$f'(\omega) = 4R(1 + R^2)\cos\theta\sin\omega - 8R^2\cos\omega\sin\omega = 0$$

if $\sin\omega = 0$ (which means $\omega = 0$ or π) or if

$$4R(1 + R)^2\cos\theta = 8R^2\cos\omega$$

which is equivalent to:

$$\cos\omega_m = \frac{1 + R^2}{2R}\cos\theta.$$

Note that we have found three critical points in the interval $[0, \pi]$, namely at the endpoints and also at the special value ω_m. We would also like to verify that ω_m is a local minimum of the function f, which would mean that it is a local maximum point for the magnitude response function $|H(\omega)|$.

First, note the values of f at the endpoints:

$$
\begin{aligned}
f(0) &= (1-R^2)^2 + 4R^2\cos^2\theta - 4R(1+R^2)\cos\theta + 4R^2 \\
&= (1-R^2)^2 + 4R^2(1+\cos^2\theta) - 4R(1+R^2)\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
f(\pi) &= (1-R^2)^2 + 4R^2\cos^2\theta + 4R(1+R^2)\cos\theta + 4R^2 \\
&= (1-R^2)^2 + 4R^2(1+\cos^2\theta) + 4R(1+R^2)\cos\theta
\end{aligned}
$$

Recall that we have f':

$$
\begin{aligned}
f'(\omega) &= 4R(1+R^2)\cos\theta\sin\omega - 8R^2\cos\omega\sin\omega \\
&= 4R(1+R^2)\cos\theta\sin\omega - 4R^2\sin(2\omega)
\end{aligned}
\tag{1}
$$

Next, we compute the second derivative of f:

$$
\begin{aligned}
f''(\omega) &= 4R(1+R^2)\cos\theta\cos\omega - 8R^2\cos(2\omega) \\
&= 4R(1+R^2)\cos\theta\cos\omega - 8R^2(2\cos^2\omega - 1)
\end{aligned}
\tag{2}
$$

Then substituting the value for $\cos\omega_m = \dfrac{1+R^2}{2R}\cos\theta$ we have:

$$
\begin{aligned}
f''(\omega_m) &= 4R(1+R^2)\cos\theta\frac{1+R^2}{2R}\cos\theta - 8R^2(2\left(\frac{1+R^2}{2R}\cos\theta\right)^2 - 1) \\
&= 2(1+R^2)^2\cos^2\theta - 4(1+R^2)^2\cos^2\theta + 8R^2 \\
&= 8R^2 - 2(1+R^2)^2\cos^2\theta \\
&= 8R^2(1 - \frac{2(1+R^2)^2}{8R^2}\cos^2\theta) \\
&= 8R^2(1 - \left(\frac{1+R^2}{2R}\right)^2\cos^2\theta) \\
&= 8R^2(1-\cos^2\omega_m) \\
&= 8R^2\sin^2\omega_m > 0
\end{aligned}
\tag{3}
$$