# Reson Filter Derivation of Peak Frequency Angle $\omega_m$ (Chapter 5.5)

Let $f(\omega)$ be the reciprocal of the magnitude response squared, written as a function of the angular frequency $\omega$. We want to find the value $\omega = \omega_m$ where $f$ achieves its maximum value on the interval $[0, \pi]$.

Then

$$\begin{aligned}
f(\omega) &= \frac{1}{|H(\omega)|^2} = |(e^{i\omega} - Re^{i\theta})(e^{i\omega} - Re^{-i\theta})|^2 \\
&= |e^{i2\omega} - Re^{i(\omega+\theta)} - Re^{i(\omega-\theta)} + R^2|^2 \\
&= |(\cos 2\omega + i \sin 2\omega) - R(\cos(\omega+\theta) + i\sin(\omega+\theta)) - R(\cos(\omega-\theta) + i\sin(\omega-\theta)) + R^2|^2 \\
&= |(\cos 2\omega - R[\cos(\omega+\theta) + \cos(\omega-\theta)] + R^2) + i(\sin 2\omega - R[\sin(\omega+\theta) + \sin(\omega-\theta)])|^2 \\
&= |(\cos 2\omega - 2R\cos\omega\cos\theta + R^2) + i(\sin 2\omega - 2R\sin\omega\cos\theta)|^2 \\
&= (\cos 2\omega - 2R\cos\omega\cos\theta + R^2)^2 + (\sin 2\omega - 2R\sin\omega\cos\theta)^2 \\
&= \cos^2 2\omega + R^4 + 4R^2\cos^2\theta\cos^2\omega + 2R^2\cos 2\omega - 4R\cos\theta\cos\omega\cos 2\omega - 4R^3\cos\theta\cos\omega \\
&\quad + \sin^2 2\omega + 4R^2\cos^2\theta\sin^2\omega - 4R\cos\theta\sin\omega\sin 2\omega \\
&= 1 + R^4 + 4R^2\cos^2\theta + 2R^2(2\cos^2\omega - 1) - 4R\cos\theta\cos\omega(1 - 2\sin^2\omega) - 4R^3\cos\theta\cos\omega \\
&\quad -8R\cos\theta\sin^2\omega\cos\omega \\
&= 1 + R^4 + 4R^2\cos^2\theta - 2R^2 + 4R^2\cos^2\omega - 4R\cos\theta\cos\omega - 4R^3\cos\theta\cos\omega \\
&= (1-R^2)^2 + 4R^2\cos^2\theta - 4R(1+R^2)\cos\theta\cos\omega + 4R^2\cos^2\omega
\end{aligned}$$

Next, we differentiate in order to find critical points for $f$:

$$f'(\omega) = 4R(1+R^2)\cos\theta\sin\omega - 8R^2\cos\omega\sin\omega = 0$$

if $\sin\omega = 0$ (which means $\omega = 0$ or $\pi$) or if

$$4R(1+R)^2\cos\theta = 8R^2\cos\omega$$

which is equivalent to:

$$\cos\omega_m = \frac{1+R^2}{2R}\cos\theta.$$