MAT 321 Homework 4
Spring 2025

Due date: Friday, March 14

Let $\mathcal{F}$ denote the Fourier transform and $\mathcal{F}^{-1}$ its inverse, so that (as in the notes of Osgood):

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt \qquad \text{and} \qquad \mathcal{F}^{-1}f(t) = \int_{-\infty}^{\infty} f(s)e^{i2\pi st}\,ds.$$

We will establish the following identities and implications in class for the continuous, infinite extent domains (SPW4). Note: we define f^- to be the function given by $f^-(t) = f(-t)$. Also, the exponent minus is assumed to apply directly to its left. If it applies to more than one symbol, we use parentheses. For example $(\mathcal{F}f)^-$ can be thought of as the function given by $g^-(s) = g(-s)$, where $g = \mathcal{F}f$.

1. $(\mathcal{F}f)^- = \mathcal{F}^{-1}f$
2. $\left(\mathcal{F}^{-1}f\right)^- = \mathcal{F}f$
3. $\mathcal{F}f^- = \mathcal{F}^{-1}f$
4. $\mathcal{F}f^- = (\mathcal{F}f)^-$
5. $\mathcal{F}^{-1}f^- = \left(\mathcal{F}^{-1}f\right)^-$
6. $\mathcal{F}^{-1}f^- = \mathcal{F}f$
7. $\mathcal{F}\mathcal{F}f = f^-$
8. f even $\iff$ $\mathcal{F}f$ even (or equivalently: f even $\iff$ $\mathcal{F}^{-1}f$ even)
9. f odd $\iff$ $\mathcal{F}f$ odd (or equivalently: f odd $\iff$ $\mathcal{F}^{-1}f$ odd)
10. f real-valued $\implies (\mathcal{F}f)^- = \overline{\mathcal{F}f} = \mathcal{F}f^-$ (same for $\mathcal{F}^{-1}$)
11. f imaginary-valued $\implies (\mathcal{F}f)^- = -\overline{\mathcal{F}f} = \mathcal{F}f^-$ (same for $\mathcal{F}^{-1}$)
12. f real-valued and even $\iff$ $\mathcal{F}f$ real-valued and even (same for $\mathcal{F}^{-1}$)
13. f imaginary-valued and even $\iff$ $\mathcal{F}f$ imaginary-valued and even (same for $\mathcal{F}^{-1}$)
14. f real-valued and odd $\iff$ $\mathcal{F}f$ imaginary-valued and odd (same for $\mathcal{F}^{-1}$)

Investigate each of these implications and identities for the case of finite domains (SPW1). In this case the time and frequency domains are still symmetric, both being finite. The variables are t and k, instead of t and s.

For each statement, first determine the analogous statement for SPW1, then decide if this new statement is true, or if it needs to be modified to make it true. For instance, you might need a factor N or $1/N$ to make it true. Finally, give the derivation of each true statement using the DFT and inverse DFT, including a factor of N or $1/N$ wherever necessary.

Note: For t values in SPW1 we assumed $0 \le t \le N-1$. Other integer t-values can also be used if we simply treat them modulo N. For example, -1 is equivalent to $N-1$ since they differ by a multiple of N. This is important in the context of the DFT since

$$e^{i\frac{2\pi}{N}k} = e^{i\frac{2\pi}{N}m}$$

whenever k and m differ by a multiple of N.