

# MAT 321 Homework 5

## Spring 2024

Due date: Friday, April 12

Let  $\mathcal{F}$  denote the Fourier transform and  $\mathcal{F}^{-1}$  its inverse, so that (as in the notes of Osgood):

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st} dt \quad \text{and} \quad \mathcal{F}^{-1}f(t) = \int_{-\infty}^{\infty} f(s)e^{i2\pi st} ds.$$

You may find portions of these problems and derivations as examples in the Osgood notes, Chapter 2-4. You may glean anything you like from there, but be sure to write up precisely only the relevant pieces for your solutions.

- Let  $\Pi(t)$  be the “rect” function, which is 1 for  $|t| < \frac{1}{2}$ , and zero elsewhere. Let  $\text{sinc}(x)$  be the function which is 1 when  $x = 0$  and is equal to  $\frac{\sin \pi x}{\pi x}$  elsewhere. Let  $\Lambda(x)$  be the function which is  $1 - |x|$  for  $|x| \leq 1$ , and zero elsewhere. Let  $f(t)$  be the one-sided exponential decay, which is  $e^{-at}$  for  $t > 0$ , and zero elsewhere. Let  $g(t)$  be the two-sided exponential decay, which is 1 when  $t = 0$ , and is  $f(t) + f(-t)$  elsewhere.

(a) Show that  $\mathcal{F}f = \frac{1}{2\pi is + a}$ .

(b) Show that  $\mathcal{F}g = \frac{2a}{a^2 + 4\pi^2 s^2}$ .

(c) Show that  $\mathcal{F}\Pi = \text{sinc}$ .

(d) Show that  $\mathcal{F}\Lambda = \text{sinc}^2$ .

(e) Let  $h(t) = \Pi(\frac{1}{2}(t - 3))$ . Find  $\mathcal{F}h$ . (Use integration by parts after splitting up the integral over two intervals. Then use the factoring trick for exponentials to get the sine function to show up.)

- Which of the functions from the previous exercise ( $f$ ,  $g$ ,  $h$ ,  $\Pi$ ,  $\text{sinc}$ ,  $\text{sinc}^2$ ,  $\Lambda$ ) are in  $L^1(\mathbb{R})$ ? Recall that a function is in  $L^1(\mathbb{R})$  if the integral of its absolute value from  $-\infty$  to  $+\infty$  converges, or exists. (see pages 137-141)
- Use duality (the types of statements in HW5 which give the various relations between the Fourier Transform, its inverse, and the minus operator) to show that:

(a)  $\mathcal{F}\text{sinc} = \Pi$ .

(b)  $\mathcal{F}\text{sinc}^2 = \Lambda$ .

- Find the convolution:  $\Pi * \Lambda$  and graph it.
- Derive the following derivative formula for functions (not distributions):

$$\mathcal{F}f'(s) = 2\pi is\mathcal{F}f(s)$$

in two different ways:

- with integration by parts, assuming both  $f$  and  $f'$  tend to zero as  $t$  approaches  $\pm\infty$ . (see page 143)
- with the definition of derivative, linearity of the Fourier Transform, the shift correspondence, and L'Hospital's Rule.