MAT 321 Homework 6
Spring 2019

Due date: Thursday, April 11

Let f be the function which is 0 for $x \le 0$, and 1 for $x \ge \frac{1}{2}$, and for $0 < x < \frac{1}{2}$ it is defined by:

$$f(x) = e^{-\frac{1}{2x} e^{\frac{1}{2x-1}}}.$$

1. For f as defined above, show the following two limits:

   a) $\lim_{x \to 0^+} f(x) = 0$

   b) $\lim_{x \to \frac{1}{2}^-} f(x) = 1$

2. Using geometric series, find a Taylor series expansion for the two rational functions at $x = a$:

   a) $-\frac{1}{2x}$ at $x = \frac{1}{2}$

   b) $\frac{1}{2x-1}$ at $x = 0$

3. (Extra Credit) Find the Laurent series expansion for $f(x)$ at both $x = 0$ and $x = \frac{1}{2}$. Use this to show that all derivatives of $f(x)$ approach zero at these two values:

   a) $\lim_{x \to 0^+} f^{(n)}(x) = 0$ for all positive integers n

   b) $\lim_{x \to \frac{1}{2}^-} f^{(n)}(x) = 0$ for all positive integers n

4. Verify by differentiating, the claim that $x \ln x - x$ is an antiderivative of $\ln x$.

5. Use L'Hospital's Rule to show that for $k > 0$: $\lim_{|x| \to 0} |x|^k \ln |x| = 0$

6. Find the limit: $\lim_{\epsilon \to 0} \int_{\epsilon}^{1} \ln x \; dx$

7. Explain why the integral $\int_{\epsilon}^{1} O(x) \ln x \; dx$ is not improper, where $O(x)$ is a function which is infinitely differentiable, and $O(0) = 0$.

8. Show that the last limit on page 210 of the Osgood Notes is zero: $\lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} \frac{\phi(x)}{x} \; dx + \int_{\epsilon}^{1} \frac{\phi(x)}{x} \; dx \right) = 0.$