MAT 321 Homework 6
Spring 2023

Due date: Wednesday, April 12

Let $\mathcal{F}$ denote the Fourier transform and $\mathcal{F}^{-1}$ its inverse, so that (as in the notes of Osgood):

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt \qquad \text{and} \qquad \mathcal{F}^{-1}f(t) = \int_{-\infty}^{\infty} f(s)e^{i2\pi st}\,ds.$$

You may find portions of these problems and derivations as examples in the Osgood notes, Chapter 2-4. You may glean anything you like from there, but be sure to write up precisely only the relevant pieces for your solutions.

1. Let $\Pi(t)$ be the "rect" function, which is 1 for $|t| < \frac{1}{2}$, and zero elsewhere. Let $sinc(x)$ be the function which is 1 when $x = 0$ and is equal to $\dfrac{\sin \pi x}{\pi x}$ elsewhere. Let $\Lambda(x)$ be the function which is $1 - |x|$ for $|x| \leq 1$, and zero elsewhere. Let $f(t)$ be the one-sided exponential decay, which is e^{-at} for $t > 0$, and zero elsewhere. Let $g(t)$ be the two-sided exponential decay, which is 1 when $t = 0$, and is $f(t) + f(-t)$ elsewhere.
   (a) Show that $\mathcal{F}f = \dfrac{1}{2\pi is + a}$.
   (b) Show that $\mathcal{F}g = \dfrac{2a}{a^2 + 4\pi^2 s^2}$.
   (c) Show that $\mathcal{F}\Pi = sinc$.
   (d) Show that $\mathcal{F}\Lambda = sinc^2$.
   (e) Let $h(t) = \Pi(\frac{1}{2}(t-3))$. Find $\mathcal{F}h$. (Use integration by parts after splitting up the integral over two intervals. Then use the factoring trick for exponentials to get the sine function to show up.)
2. Which of the functions from the previous exercise (f, g, h, Π, $sinc$, $sinc^2$, Λ) are in $L^1(\mathbb{R})$? Recall that a function is in $L^1(\mathbb{R})$ if the integral of its absolute value from $-\infty$ to $+\infty$ converges, or exists. (see pages 137-141)
3. Use duality (the types of statements in HW5 which give the various relations between the Fourier Transform, its inverse, and the minus operator) to show that:
   (a) $\mathcal{F}sinc = \Pi$.
   (b) $\mathcal{F}sinc^2 = \Lambda$.
4. Find the convolution: $\Pi * \Lambda$ and graph it.
5. Derive the following derivative formula for functions (not distributions):

$$\mathcal{F}f'(s) = 2\pi is\mathcal{F}f(s)$$

   in two different ways:
   (a) with integration by parts, assuming both f and f' tend to zero as t approaches $\pm\infty$. (see page 143)
   (b) with the definition of derivative, linearity of the Fourier Transform, the shift correspondence, and L'Hospital's Rule.