MAT 351 Homework 1

Summer 2019

Due Date: Wednesday, May 15

For the group theory problems you may use the following facts: (note: the group operation is written as concatenation or $*$)

- In any group G, the identity and inverses are unique.
- In any group G, the cancellation law holds: for $a, b, c \in G$ if $ab = ac$ or $ba = ca$, then we can conclude that $b = c$.
- For x in any group G, let $ord(x)$ denote the smallest positive integer k such that $x^k = e$, where $x^k = x * x * \cdots * x$ means x combined (multiplied) with itself (k factors) using the binary operation of G. Thus $ord(e) = 1$ and $ord(x) \geq 2$ for all other elements of G.
- It is a fact that the orders of elements are preserved by an isomorphism, and the order of any element in a group must be a divisor of the number of elements in the group.
- If H is any subgroup of G, then the order of H must be a divisor of the order of G. (Lagrange's Theorem)
- The set of permutations (one-to-one functions) of the set $\{1, 2, \ldots, n\}$ to itself is a group, with composition as the binary operation, and there are a total of $n!$ elements in this group.

1. Find the orthogonal 3×3 matrix which represents rotation around the axis which points in the direction of $(2, 1, 1)$ and rotates an angle of $\pi/3$ radians counterclockwise. Represent the numbers algebraically or numerically to 2 decimal places. Find the matrix B which can be used to extract the axis of rotation and verify that this works. Verify that the angle can be recovered from the trace formula.
2. Construct a regular tetrahedron out of cardboard, find the group of rotational symmetries, and write down at least one fourth of the entries in the group table. Note: The tetrahedron has four sides, each being an equilateral triangle of the same size. The rotational symmetries are rotations which preserve the original shape, but can switch the vertices around. (You may use permutation notation by numbering the vertices if you like.)
3. Show that there is only one possible group of order three up to isomorphism, by arguing that only one group table can be constructed. (Think of the process of filling in a 3×3 group table as forming a decision tree. At each node you should have either a complete and valid table, or a conclusion that there cannot be any group table based on the given partial table construction.)
4. Write the group table for Q_8, the finite quaternion group given the following definitions: $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, where the group elements are abbreviations for 2×2 matrices with entries in the complex numbers (matrix entry $i = \sqrt{-1}$) as follows:

$$\pm\mathbf{1} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\mathbf{i} = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm\mathbf{j} = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm\mathbf{k} = \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and the group operation is matrix multiplication. Note: the symbol i is overloaded, but you should be able to use it carefully based on context.