MAT 351 Homework 2

Summer 2019

Due Date: Thursday, May 23

1. Find a subgroup of $GL_2(\mathbf{F}_3)$ which is isomorphic to Q_8. The set $\mathbf{F}_3$ is the finite field with 3 elements $\{0, 1, 2\}$. All operations are done mod 3, including addition, multiplication, and division (which is just multiplication by the inverse). The group $GL_n(\mathbf{F})$ for any field $\mathbf{F}$ is the group of nonsingular $n \times n$ matrices with entries in the field $\mathbf{F}$. Show all work used to verify that this subgroup is isomorphic to Q_8. (This may also be code plus output.)

2. Recall that two finite groups are isomorphic if their group tables are essentially the same, up to reordering and relabeling of the elements. Show that U_4 and $\mathbf{Z}_2 \times \mathbf{Z}_2$ are not isomorphic, where U_4 denotes the group of complex fourth roots of unity: $\{1, -1, i, -i\}$ with binary operation as multiplication.

3. Find all cyclic subgroups of Q_8, the finite quaternion group. (Recall: a cyclic group must be generated by one element and all of its multiples, or powers, using the binary operation of the group.)

4. Let G be a finite group, and S be a subset of G, and let $\langle S \rangle$ be the smallest subgroup of G containing S. Let H be the collection of products $x_1 x_2 \cdots x_k$, where all of the x_i are in the union of sets $S \cup S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$, ie. $H = \{x_1 x_2 \cdots x_k : x_1, x_2, \ldots, x_k \in S \cup S^{-1}\}$. Show that H is a subgroup of G using the criterion: "H is a subgroup of G if and only if whenever x and y are in H, then xy^{-1} is in H." Then show that $H = \langle S \rangle$. (H is called the subgroup of G generated by S.)

5. Recall: A presentation of a group G is written as: $G = \langle a_1, a_2, a_3, \ldots, a_m : E_1, E_2, E_3, \ldots, E_n \rangle$ meaning that G has elements a_i which satisfy the equations, or relations, E_i. The size of the presentation is the sum $m + n$.

   (a) Find a presentation of the quaternion group Q_8 with exactly four generators and eight relations.

   (b) Give reasons why the presentation in the previous part generates Q_8.

   (c) Give reasons why at least one generator and one relation in this presentation are not needed to generate all of Q_8.

   (d) Give a smaller presentation based on the previous two parts, by eliminating at least one generator and one relation.