MAT 351 Homework 3 - Summer 2019

Due: Thursday, May 30, 2019

Notation:

- S^1 = the unit circle in the complex plane, or equivalently, those complex numbers of length one. Polar form can be written as $e^{i\theta}$ ($r = 1$.)
- $SO(2) = 2 \times 2$ rotation matrices.
- Correspondence between S^1 and $SO(2)$:

$$e^{i\theta} \leftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1. Let F be the set $\mathbb{R}^2$ with addition $(a,b)+(c,d) = (a+c, b+d)$ and multiplication $(a,b)\cdot(c,d) = (a\cdot c, b\cdot d)$. Show that F is not a field, even though it has both additive and multiplicative identities ($(0,0)$ and $(1,1)$) and its entries come from the field of real numbers.

2. Show that complex numbers w and z are linearly dependent (as vectors) if and only if $w\bar{z} \in \mathbb{R}$. (Note: the linear dependence statement uses only the vector space properties of $\mathbb{C}$, but the criterion in this case uses the multiplication of $\mathbb{C}$.)

3. Prove that three numbers a, b, and $c \in \mathbb{C}$ are collinear if and only if $\dfrac{c-a}{b-a} \in \mathbb{R}$, that is, if and only if $c\bar{b} - c\bar{a} - a\bar{b} \in \mathbb{R}$. (Hint: Use the paramtric form of a line.)

4. Find the smallest counterexample to the (False) "Three-Squares Theorem" which states "Any number which factors as the product of two sums of three squares must itself be a sum of three squares." (By smallest we mean using the smallest possible whole numbers. Also, the statement is meant to imply that the formula involves polynomial expressions only, just as in the two-squares theorem. So if the number is $N = (a^2+b^2+c^2)(x^2+y^2+z^2)$ then we should also have $N = u^2+v^2+w^2$, where u, v and w can be written in terms of polynomial expressions involving only a,b,c, x,y and z and integer coefficients. Thus, if we can find N and a,b,c,x,y and z, but can show that there does not exist whole numbers u, v and w with $N = u^2+v^2+w^2$, we have a contradiction to the "Three-Squares Theorem".)

5. Let L denote a line in the complex plane which passes through the origin. Assume that L is not the x-axis. Show that the points on the line L, as a subset of $\mathbb{C}$, cannot be a subfield of $\mathbb{C}$.

6. (a) Find a parametrization of the shortest arc on S^1 from the point $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ to the point -1. Let z_t be the parametrization, with $0 \le t \le 1$. Transfer this to a parametrization A_t between the two corresponding matrices in $SO(2)$.

   (b) Find a parametrization of the longer arc on S^1 from the point $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ to the point -1. Let z_t' be this parametrization, with $0 \le t \le 1$. Transfer this to a parametrization A_t' between the two corresponding matrices in $SO(2)$.

7. (a) Write down an algorithm which takes as input two points in S^1 in Cartesian form, and gives as output the shortest path from one point to the other on the unit circle. The path should be a parametrization (as in the previous problem) with parameter t, and the output written in polar form. Say exactly how you are using the polar form and how you choose the angles to represent each of the complex numbers.

   (b) Use your algorithm in the previous part applied to three different examples of your own choice. Make the examples diverse enough to perform a good test of the correctness of your algorithm.