MAT 351 Homework 4 - Summer 2019

Due: Thursday, June 6, 2019

1. Let $q = a + bi + cj + dk \in S^3$, $p = w + xi + yj + zk \in \mathbb{H}$, $f_q(p) = qp$, $g_{\bar{q}}(p) = p\bar{q}$, and $R_q(p) = qp\bar{q}$. Also, let $\mathbf{v} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4$ and let T be the function which translates a quaternion into a vector, so that $T(p) = \mathbf{v}$ for instance.

   (a) Find a 4×4 real matrix A_q such that $A_q\mathbf{v} = T(f_q(p))$.

   (b) Find a 4×4 real matrix B_q such that $B_q\mathbf{v} = T(g_{\bar{q}}(p))$.

   (c) Find a 4×4 real matrix C_q such that $C_q\mathbf{v} = T(R_q(p))$.

2. Let $C = (\mathbb{C}, *)$ be the algebra with set of elements coinciding with the complex numbers $\mathbb{C}$ but with multiplication * given by: $w * z = \overline{wz}$. Show that C is a non-associative, 2-dimensional, division algebra, without multiplicative identity.

3. Classify the following subsets of $\mathbb{H}$ according to their substructures: Answer TRUE or FALSE to each statement about S, and briefly explain. (Note: the symbols i, j, k are reserved for the special unit quaternions, q a general quaternion, $\mathbf{v}$ a pure imaginary quaternion, and the symbols a, b, c, d will be real numbers, or scalars.)

   i) S is a sub vector space of $\mathbb{H}$.

   ii) S is s subalgebra of $\mathbb{H}$.

   iii) S is a subgroup of $\mathbb{H}^*$.

   (a) $S = \{a + bk : a, b \in \mathbb{R}\}$.

   (b) $S = \{bi + cj + dk : b^2 + c^2 + d^2 = 1\}$.

   (c) $S = \{q \in \mathbb{H} \ : |q| = 2\}$.

   (d) $S = \{q \in \mathbb{H} \ : q \bullet (i + j + k) = 0\}$.

   (e) $S = \{1, i, j, k\}$.

   (f) $S = \{\pm 1, \pm i, \pm j, \pm k\}$.

4. Let p and q be any elements of the quaternion algebra $\mathbb{H}$. Show that the following are equivalent:

   (a) $pq = qp$

   (b) 1, p, and q are linearly dependent (considered as vectors in $\mathbb{R}^4$)

   (c) There exists a 2-dimensional subalgebra of $\mathbb{H}$, isomorphic to $\mathbb{C}$(as an algebra, hence also as a field) containing p and q. Hint: The subalgebra can be defined using a basis product table.