1. Let $q = a + bi + cj + dk \in S^3$, $p = w + xi + yj + zk \in \mathbb{H}$, $f_q(p) = qp$, $g_q(p) = p\bar{q}$, and $R_q(p) = qp\bar{q}$. Also, let $v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4$ and let $T$ be the function which translates a quaternion into a vector, so that $T(p) = v$ for instance.

(a) Find a $4 \times 4$ real matrix $A_q$ such that $A_qv = T(f_q(p))$.
(b) Find a $4 \times 4$ real matrix $B_q$ such that $B_qv = T(g_q(p))$.
(c) Find a $4 \times 4$ real matrix $C_q$ such that $C_qv = T(R_q(p))$.

2. Let $C = (\mathbb{C}, *)$ be the algebra with set of elements coinciding with the complex numbers $\mathbb{C}$ but with multiplication $*$ given by: $w * z = \overline{wz}$. Show that $C$ is a non-associative, 2-dimensional, division algebra, without multiplicative identity.

3. Classify the following subsets of $\mathbb{H}$ according to their substructures: Answer TRUE or FALSE to each statement about $S$, and briefly explain. (Note: the symbols $i, j, k$ are reserved for the special unit quaternions, $q$ a general quaternion, $v$ a pure imaginary quaternion, and the symbols $a, b, c, d$ will be real numbers, or scalars.)

i) $S$ is a sub vector space of $\mathbb{H}$.
ii) $S$ is a subalgebra of $\mathbb{H}$.
iii) $S$ is a subgroup of $\mathbb{H}^*$.

(a) $S = \{a + bk : a, b \in \mathbb{R}\}$.
(b) $S = \{bi + cj + dk : b^2 + c^2 + d^2 = 1\}$.
(c) $S = \{q \in \mathbb{H} : |q| = 2\}$.
(d) $S = \{q \in \mathbb{H} : q \cdot (i + j + k) = 0\}$.
(e) $S = \{1, i, j, k\}$.
(f) $S = \{\pm 1, \pm i, \pm j, \pm k\}$.

4. Let $p$ and $q$ be any elements of the quaternion algebra $\mathbb{H}$. Show that the following are equivalent:

(a) $pq = qp$
(b) 1, $p$, and $q$ are linearly dependent (considered as vectors)
(c) There exists a 2-dimensional subalgebra of $\mathbb{H}$, isomorphic to $\mathbb{C}$, (as an algebra, hence also as a field) containing $p$ and $q$. 