MAT 351 Homework 5 - Summer 2019

Due: Friday, June 28, 2019

1. Show that $Slerp(p, q, t)$ can also be given by the linear combination of p and q: (where θ is the angle between p and q.)
$$Slerp(p, q, t) = \frac{\sin(1-t)\theta}{\sin\theta} p + \frac{\sin t\theta}{\sin\theta} q.$$

2. Prove the triple product identity for quaternions: For any $p, q \in \mathbb{H}$:
$$qpq = 2(\bar{p} \bullet q)q - (q \bullet q)\bar{p}.$$
(Hints: Start with the fact that for any quaternion x: $x\overline{x} = x \bullet x$, (call this equation (1)), since both are equal to $|x|^2$. Now replace x with $p + q$ in equation (1) and multiply out both sides using distributive laws for dot and quaternion products, and commutative law for dot (but not quaternion) product. Solve for $2p \bullet q$ and call this equation (2). Now mutiply by q on the right on both sides of (2) and solve for $q\overline{p}q$. Call this equation (3). Finally, substitute p with $\overline{p}$ in equation (3) to get the final result.)

3. Show that the imaginary quaternion space can be described as:
$$Im(\mathbb{H}) = \{q \in \mathbb{H} : q^2 \in Re(\mathbb{H}) \ and \ q \notin Re(\mathbb{H})\backslash\{0\}\}$$

4. Prove that if $q \in S^3$ is a unit quaternion, then the operator $R_q(\mathbf{v}) = q\mathbf{v}\bar{q}$ for imaginary quaternions $\mathbf{v} \in Im(\mathbb{H})$ performs the rotation about an axis and angle specified in the polar form of q. (ie. if $q = \cos\theta + \sin\theta\mathbf{u}$ then the axis is $\mathbf{u}$ and the angle is 2θ.) Do this by working directly with the special product formula and comparing to the rotation formula:
$$R\mathbf{v} = (1 - \cos\theta)(\mathbf{v} \bullet \mathbf{u})\mathbf{u} + \cos\theta\mathbf{v} + \sin\theta\mathbf{u} \times \mathbf{v}.$$

5. (a) Find the rotational group G of symmetries of a cube in $\mathbb{R}^3$. Write the elements by specifying axis and angle of rotation, counterclockwise about the axis, using the right hand rule, in radians. (Use the vertices of the cube: $(\pm1, \pm1, \pm1)$. For conveience, you may use non-unit vectors as labels for axes, even though any implementation would use a unit vector.)

   (b) Show that the previous group is isomorphic to the permutation group S_4 by considering how the rotations affect the four long diagonals of the cube.

   (c) Find the preimage of G under the 2 to 1 mapping of S^3 to $SO(3)$. Note: The preimage of a set T in the range of a function f is the set of elements S is the domain of f such that for all x in S, $f(x)$ is in T. Find each element of the preimage as a unit quaternion in both rectangular and polar forms. Call this preimage set H. Show that H is a group of order 48.

   (d) Determine if H is isomorphic, or not, to $GL_2(\mathbf{F}_3)$ (which also has order 48.)