1. Show that $\text{Slerp}(p,q,t)$ can also be given by the linear combination of $p$ and $q$: (where $\theta$ is the angle between $p$ and $q$.)

$$\text{Slerp}(p,q,t) = \frac{\sin(1-t)\theta}{\sin \theta} p + \frac{\sin t \theta}{\sin \theta} q.$$

2. Prove the triple product identity for quaternions: For any $p,q \in \mathbb{H}$:

$$pq = 2(p \cdot q)q - (q \cdot q)p.$$

(Hints: Start with the fact that for any quaternion $x$: $x\overline{x} = x \cdot x$, (call this equation (1)), since both are equal to $|x|^2$. Now replace $x$ with $p + q$ in equation (1) and multiply out both sides using distributive laws for dot and quaternion products, and commutative law for dot (but not quaternion) product. Solve for $2p \cdot q$ and call this equation (2). Now multiply by $q$ on the right on both sides of (2) and solve for $q\overline{q}p$. Call this equation (3). Finally, substitute $p$ with $p$ in equation (3) to get the final result.)

3. Show that the imaginary quaternion space can be described as:

$$\text{Im} (\mathbb{H}) = \{ q \in \mathbb{H} : q^2 \in \text{Re} (\mathbb{H}) \text{ and } q \notin \text{Re} (\mathbb{H}) \backslash \{0\} \}$$

4. Prove that if $q \in S^3$ is a unit quaternion, then the operator $R_q(v) = qv\overline{q}$ for imaginary quaternions $v \in \text{Im}(\mathbb{H})$ performs the rotation about an axis and angle specified in the polar form of $q$. (ie. if $q = \cos \theta + \sin \theta u$ then the axis is $u$ and the angle is $2\theta$.) Do this by working directly with the special product formula and comparing to the rotation formula:

$$Rv = (1 - \cos \theta)(v \cdot u)u + \cos \theta v + \sin \theta u \times v.$$

5. (a) Find the rotational group $G$ of symmetries of a cube in $\mathbb{R}^3$. Write the elements by specifying axis and angle of rotation, counterclockwise about the axis, using the right hand rule, in radians. (Use the vertices of the cube: $(\pm 1, \pm 1, \pm 1)$. For convenience, you may use non-unit vectors as labels for axes, even though any implementation would use a unit vector.)

(b) Show that the previous group is isomorphic to the permutation group $S_4$ by considering how the rotations affect the four long diagonals of the cube.

(c) Find the preimage of $G$ under the 2 to 1 mapping of $S^3$ to $SO(3)$. Note: The preimage of a set $T$ in the range of a function $f$ is the set of elements $S$ is the domain of $f$ such that for all $x$ in $S$, $f(x)$ is in $T$. Find each element of the preimage as a unit quaternion in both rectangular and polar forms. Call this preimage set $H$. Show that $H$ is a group of order 48.

(d) Determine if $H$ is isomorphic, or not, to $GL_2(\mathbb{F}_3)$ (which also has order 48.)