MAT 351 Supplementary Notes Spring 2013

Quaternion Interpolation

(Note: These notes are meant to supplement the lectures. Full lecture notes can only be obtained by attending class lectures and taking notes.)

Interpolation with Unit Quaternions

Orientations in 3D

An 3D object can be thought of as a set of points in $\mathbb{R}^3$. If we fix the axes x,y,z in some configuration, then the object will have some original or default position in space. Now if we apply a transformation to all the points in $\mathbb{R}^3$ then the object is moved to a new position. If the transformation is an orthogonal one, then we say that the object has undergone a *rigid* transformation. If the transformation is orthogonal and also has determinant one, ie. it is a rotation, then we say the transformed object has a new *orientation.* In this way, any rotation corresponds to an orientation of the object. If we want to choose a sequence of orientations of an object, we can specify a sequence of matrices $\in SO(3)$. Each matrix applied to $\mathbb{R}^3$ as a transformation will yield the new orientation of the object.

Since we know that the unit quaternions are a double cover of $SO(3)$, we can associate to any sequence $M_0, M_1, \ldots, M_n$ of rotation matrices, a sequence of pairs of unit quaternions $(q_0, -q_0), (q_1, -q_1), \ldots, (q_n, -q_n)$. We state the basic interpolation problem:

Definition: For any sequence $M_0, M_1, \ldots, M_n$ of rotation matrices, we call an interpolating curve $M(t)$ of matrices, a continuous function from $\mathbb{R}$ to $SO(3)$ such that for some values $t_0 < t_1 < \cdots < t_n$, we have $M(t_i) = M_i$.

Definition: For any sequence $q_0, q_1, \ldots, q_n$ of unit quaternions, we call an interpolating curve $q(t)$ of unit quaternions, a continuous function from $\mathbb{R}$ to S^3 such that for some values $t_0 < t_1 < \cdots < t_n$, we have $q(t_i) = q_i$.

Circles in $\mathbb{R}^4$

Here we set up the neccessary prerequisite material for the interpolation methods that follow. The most basic geometric construction is the great arc on S^3. We will also use other circles on S^3 (of radius smaller than 1) in such methods as the Circular Blending technique (of Kim et al.).

We refer to points in $\mathbb{R}^4$ as P, Q, etc. and we will also interchangeably refer to vectors $\mathbf{u}$ and $\mathbf{v}$ in the vector space $\mathbb{R}^4$. In each case, the point or the vector is uniquely determined by its Cartesian coordinates (w, x, y, z).

Definition: A *great circle* in $\mathbb{R}^4$ is the set of all points (w, x, y, z) which lie on both the unit 3-sphere S^3, and on a 2-plane through the origin. We also call this the *intersection* of S^3 and a 2-plane through the origin.

Definition: A 2-plane through the origin in $\mathbb{R}^4$ is simply a 2-dimensional subspace of $\mathbb{R}^4$, where we identify points with column vectors. Specifically, if $\mathbf{u}$ and $\mathbf{v}$ are any two linearly independent vectors, then the span of $\mathbf{u}$ and $\mathbf{v}$, ie. the set of all linear combinations $a\mathbf{u} + b\mathbf{v}$ is such a 2-plane.

Definition: A circle in $\mathbb{R}^4$ is the result of transforming any great circle by scaling and translation. This can be written as the set of points $(w, x, y, z) = (c_1, c_2, c_3, c_4) + r(x_1, x_2, x_3, x_4)$, where (c_1, c_2, c_3, c_4) is the center, r the radius, and (x_1, x_2, x_3, x_4) represents any point on the great circle.

Lemma: Parametric form: Any Circle in $\mathbb{R}^4$ can be written in parametric form as: $(w, x, y, z) = \mathbf{c} + \cos t\mathbf{u} + \sin t\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal 4-vectors with the same length $r = |\mathbf{u}| = |\mathbf{v}|$. In particular, if $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{c} = \mathbf{0}$, then we have a great circle.

Definition: A *great arc* in $\mathbb{R}^4$ is a piece of a great circle. More specifically, a great arc between two points P and Q on S^3 is a piece of the great circle containing P and Q with endpoints at P and Q.

SLERP: Spherical Linear Interpolation

If $\mathbf{u}$ and $\mathbf{w}$ are unit vectors, we can write $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1$ parallel to $\mathbf{u}$ and $\mathbf{w}_2$ orthogonal to $\mathbf{u}$, by taking $\mathbf{w}_1 = proj_{\mathbf{u}}(\mathbf{w})$ and $\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1$. We call $\mathbf{w}_1$ the vector component of $\mathbf{w}$ parallel to $\mathbf{u}$, and $\mathbf{w}_2$ the vector component of $\mathbf{w}$ orthogonal to $\mathbf{u}$. We will also call $\frac{\mathbf{w}_2}{|\mathbf{w}_2|}$ the *normalized vector component of* $\mathbf{w}$ *orthogonal to* $\mathbf{u}$. If $\mathbf{u}$ and $\mathbf{w}$ are not neccessarily unit vectors, we also define:

Definition: If $\mathbf{u}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and $\mathbf{u} \bullet \mathbf{w} \neq 0$, then we define $\mathbf{w}_{\mathbf{u}}^{\perp}$ to be the vector component of $\mathbf{w}$ orthogonal to $\mathbf{u}$, normalized to have the same length as $\mathbf{u}$, ie.:

$$\mathbf{w}_{\mathbf{u}}^{\perp} = |\mathbf{u}| \frac{\mathbf{w} - proj_{\mathbf{u}}\mathbf{w}}{|\mathbf{w} - proj_{\mathbf{u}}\mathbf{w}|}.$$

Lemma: Let P and Q be points on S^3. Then the parametric form of the shortest great arc from P to Q is achieved as follows: Let $\mathbf{u}$ and $\mathbf{w}$ be vectors with endpoints at P and Q respectively. Then let $\mathbf{v}$ be the normalized vector component of $\mathbf{w}$ orthogonal to $\mathbf{u}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{w}$. Then the shortest great arc from P to Q can be written as: $(w, x, y, z) = \cos t\mathbf{u} + \sin t\mathbf{v}$, $0 \leq t \leq \theta$.

Lemma: Parametric form of the complementary great arc from P to Q: Let $\mathbf{u}$ and $\mathbf{w}$ be vectors positioned at P and Q respectively. Then let $\mathbf{v}$ be the normalized vector component of $\mathbf{w}$ orthogonal to $\mathbf{u}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{w}$. Then the complementary great arc from P to Q can be written as: $(w, x, y, z) = \cos t\mathbf{u} - \sin t\mathbf{v}$, $0 \leq t \leq 2\pi - \theta$.

Definition: Spherical Linear Interpolation, or $Slerp(P, Q, t)$ is the name given to a path in S^3 which goes from a point P to a point Q along the shortest great arc from P to Q.

Lemma: Slerp (Basic Circular Form): For p and q unit quaternions in S^3 and let $\mathbf{u}$ and $\mathbf{w}$ be 4-vectors representing p and q respectively. Then let $\mathbf{v}$ be the normalized vector component of $\mathbf{w}$ orthogonal to $\mathbf{u}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{w}$. Then

$$(w, x, y, z) = \cos t\mathbf{u} + \sin t\mathbf{v}, \ 0 \leq t \leq \theta.$$

Lemma: Quaternion Slerp: For p and q unit quaternions in S^3

$$Slerp(p, q, t) = p(p^{-1}q)^t.$$

We verify the above first for the path from 1 to q. Then the above formula reduces to $Slerp(1, q, t) = q^t$. In order to show that this is the required path, we first show that the vectors associated to the quaternions 1, q, and q^t, are linearly dependent for any t.

Quaternion Curves in S^3

Definition: A quaternion curve will denote a continuous function $q(t)$, $a \leq t \leq b$, such that $q(t) \in S^3$.

Squad: Spherical Quadrangle Interpolation

The curve called Squad is a generalization of the parabolic blending method from $\mathbb{R}^n$. It is easiest to describe this in $\mathbb{R}^2$. First construct a quadrilateral with corner points P_0, P_1, P_2, and P_3, and such that a path around the boundary can be described in the order given (so P_2 is farthest from P_0 if we travel around the boundary). The parabolic blend is a curve that starts at P_0 and ends at P_3. To define it, we interpolate linearly along two lines simultaneously from P_0 to P_3 as well as from P_1 to P_2. Denote each of these as $lin(P_0, P_3, t)$ and $lin(P_1, P_2, t)$. So we have:

$$lin(P, Q, t) = (1 - t)P + tQ.$$

Finally, we use a quadratic blending parameter between these two linear interpolations to obtain the parabolic blend (PB) as:

$$PB(t) = lin(lin(P_0, P_3, t), lin(P_1, P_2, t), 2t(1 - t)).$$

In the special case: $P_0 = (0, 0)$, $P_1 = (0, 1)$, $P_2 = (1, 1)$, and $P_3 = (1, 0)$, we have the unit square as our quadrilateral, and the parabolic blend $PB(t)$ becomes simply the graph of the function $f(t) = 2t(1 - t)$. This parabola reaches a maximum of $\frac{1}{2}$ at $t = \frac{1}{2}$.

The quaternion analog of $PB(t)$ is obtained by replacing linear interpolation with SLERP. The resulting curve is called Squad for “spherical quadrangle”:

$$Squad(t) = Slerp(Slerp(q_0, q_3, t), Slerp(q_1, q_2, t), 2t(1 - t)).$$

Shoemake’s Bezier Curve Interpolation

This is the analog of the DeCastlejau Algorithm in $\mathbb{R}^n$ extended to the unit quaternion sphere S^3. In particular, Shoemake uses the three step recursion, or cubic Bezier curve to obtain a C^1 interpolation curve on S^3. The actual curve is no longer cubic, ie. made up of degree 3 polynomials, but it is sometimes still referred to as cubic. The three linear interpolations are replaced with Slerp on S^3. In the following formula, q_0 and q_3 are the first and last points of the curve, ie. $q(0) = q_0$ and $q(1) = q_3$. The points q_1 and q_2 are control points (or control quaternions).

$$\begin{aligned} ShoBez(t) &= Slerp(Slerp(Slerp(q_0, q_1, t), Slerp(q_1, q_2, t), t), \\ &\quad Slerp(Slerp(q_1, q_2, t), Slerp(q_2, q_3, t), t), t) \end{aligned}$$

Lemma: The derivative $q'(0)$ of $q(t) = ShoBez(t)$ is a scalar multiple of the derivative $r'(0)$ of $r(t) = Slerp(q_0, q_1, t)$

These curves can be used for interpolation of a sequence of key quaternions $q_0, q_1, \ldots, q_n$. The question is how to place control quaternions in order to achieve smoothness at the key points. Shoemake’s idea was to place the control quaternions along joining great arcs across the key points. Suppose we insert control points as follows:

$$q_0, a_0, b_0, q_1, a_1, b_1, \ldots, q_{n-1}, a_{n-1}, b_{n-1}, q_n.$$

Shoemake’s formula for the a_i and b_i: Let $r_q(p) = 2(p \bullet q) - p$. Let

$$Double(p, q) = 2(p \bullet q) - p,$$

and

$$Bisect(p, q) = \frac{p + q}{|p + q|}.$$

Then define a_n and b_n as:

$$a_n = Bisect(Double(q_{n-1}, q_n), q_{n+1}), \quad b_n = Double(a_n, q_n).$$

A Parametric Adjustment to Shoemake Bezier

Implementations of Shoemakes methods applied to a sequence of orientations of an object show that the method often suffers from excessive or less than optimal path length and curvature. An easy way to adjust for this problem is to add a parameter s which shortens the length of the circular arcs between the key and control quaternions.

Each quaternion curve in Shoemake's scheme is constructed with four control quaternions, just as a cubic Bezier curve has four control points, and two of these quaternions are on the curve and the other two are not. For example, in the first curve we have control

Cumulative Form for Quaternion Bezier and B-Spline Curves

Quadratic Bernstein Polynomials: $B_0(t) = (1-t)^2$, $B_1(t) = 2t(1-t)$, $B_2(t) = t^2$.

Cumulative Form: $C_0(t) = B_0(t) + B_1(t) + B_2(t) = 1$, $C_1(t) = B_1(t) + B_2(t) = 1 - B_0(t) = 1 - (1-t)^2$, $C_2(t) = B_2(t) = t^2$.

Quadratic Cumulative Bezier Curve:

$$\begin{aligned} CumBez(t) &= q_0^{B_0(t)}(q_0^{-1}q_1)^{B_1(t)}(q_1^{-1}q_2)^{B_2(t)} \\ &= q_0(q_0^{-1}q_1)^{1-(1-t)^2}(q_1^{-1}q_2)^{t^2}. \end{aligned}$$

Proposition If q_1 is on the great arc joining q_0 and q_2, then the path $CumBez(t)$ defined above is along the same great arc.

Proof: Write $q_1 = q_0(q_0^{-1}q_2)^\alpha$. Then we can write the inverse as $q_1^{-1} = (q_0^{-1}q_2)^{1-\alpha}q_2^{-1}$. This follows by checking the multiplication (and applying the lemma)

$$\begin{aligned} q_1q_1^{-1} &= q_0(q_0^{-1}q_2)^\alpha(q_0^{-1}q_2)^{1-\alpha}q_2^{-1} \\ &= q_0(q_0^{-1}q_2)^{\alpha+1-\alpha}q_2^{-1} \\ &= q_0(q_0^{-1}q_2)q_2^{-1} \\ &= (q_0q_0^{-1})(q_2q_2^{-1}) \\ &= 1 \end{aligned}$$

Now we substitute into $CumBez(t)$ to obtain:

$$\begin{aligned} CumBez(t) &= q_0(q_0^{-1}q_1)^{1-(1-t)^2}(q_1^{-1}q_2)^{t^2} \\ &= q_0(q_0^{-1}q_0(q_0^{-1}q_2)^\alpha)^{1-(1-t)^2}((q_0^{-1}q_2)^{1-\alpha}q_2^{-1})q_2)^{t^2} \\ &= q_0((q_0^{-1}q_2)^\alpha)^{1-(1-t)^2}((q_0^{-1}q_2)^{1-\alpha})^{t^2} \\ &= q_0(q_0^{-1}q_2)^{\alpha(1-(1-t)^2)+(1-\alpha)t^2} \qquad (1) \\ &= q_0(q_0^{-1}q_2)^{2\alpha t+(1-2\alpha)t^2}. \end{aligned}$$

This last expression is on the great circle containing q_0 and q_2.

Exercise: Show that $ShoBez(t) \neq CumBez(t)$ for degree 2 and degree 3.

Circular Arc Blending

Here we follow the method of Kim et al. but with a slightly different development. We focus on the elementary parametrization of circles.

Given a sequence of unit quaternions $q_1, \ldots, q_n$, we construct a path $B(t)$ which interpolates the q_i by blending circular arcs. First, through any three consecutive distinct points (quaternions q_{i-1}, q_i, and q_{i+1}) there exists a unique circle in $\mathbb{R}^4$. The center of the circle can be found with vector geometry, and thus we can write a parametrization for the circular arc from q_{i-1} to q_{i+1} passing through q_i, and call it $C_i(t)$. Suppose that $C_i(0) = q_{i-1}$, $C_i(\alpha_i) = q_{i-1}$, and $C_i(\beta_i) = q_{i-1}$. Between any two quaternions q_i and q_{i+1} there are now two circular paths: $C_i(t)$ and $C_{i+1}(t)$. Next, we reparametrize these two paths so that we have $C_i(s)$ and $C_{i+1}(s)$, $0 \le s \le 1$, with $C_i(0) = C_{i+1}(0) = q_i$ and $C_i(1) = C_{i+1}(1) = q_{i+1}$. Finally, we define $B_i(s) = Slerp(C_i(s), C_{i+1}(s), s)$. Piecing all the B_i together will give a path $B(t)$ as desired. We call this ultimate path the circular blending quaternion interpolation curve.

Details:

1. First we find the center of the circle. For simplicity, let $i = 1$. Any 3 distinct unit quaternions q_0, q_1, and q_2, are noncolliner, so they determine a circle, and hence also a plane, call it M, in $\mathbb{H}$. Let M_1 and M_2 be the midpoints of the line segments from q_0 to q_1 and q_1 to q_2 respectively. Also let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the vectors from q_0 to q_1 and q_1 to q_2 respectively. Further, let $\mathbf{w}_1$ be the vector component of $\mathbf{v}_1$ orthogonal to $\mathbf{v}_2$ and let $\mathbf{w}_2$ be the vector component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$. It is then possible to parametrize the perpendicular bisector (in the plane M) of the line segment from q_0 to q_1 with the midpoint M_1 and the perpendicular vector $\mathbf{w}_2$. The other line bisector can be parametrized similarly. Finally, the center C of the circle is just the intersection of these two bisectors.
2. The parametrization $C_1(t)$ can be written as

$$C_1(t) = C + \cos t\mathbf{u} + \sin t\mathbf{v},\ 0 \le t \le \theta$$

for some appropriately chosen $\mathbf{u}$, $\mathbf{v}$ and θ. First, let $\mathbf{u}$ be the vector from C to q_0. Next, we will choose $\mathbf{v}$ to be one of the two vectors parallel to M, with length $|\mathbf{u}|$, and orthogonal to $\mathbf{u}$. In order to determine which one, let $\mathbf{u}_1$ be vector from C to q_1, and let $\mathbf{u}_2$ be vector from C to q_2. Since q_0, q_1, and q_2 are distinct points on a circle, it is possible for at most one of $\mathbf{u}_1$ or $\mathbf{u}_2$ to be opposite $\mathbf{u}$. When either (or both) of these vectors are not opposite $\mathbf{u}$, we define $\mathbf{w}_1$ and $\mathbf{w}_2$ to be the vector components of $\mathbf{u}_1$ and $\mathbf{u}_2$ orthogonal to $\mathbf{u}$, normalized to have the same length as $\mathbf{u}$. That is: $\mathbf{w}_1 = (\mathbf{u}_1)^{\perp}_{\mathbf{u}}$ and $\mathbf{w}_2 = (\mathbf{u}_2)^{\perp}_{\mathbf{u}}$. Let α be the angle between $\mathbf{u}$ and $\mathbf{u}_2$. Now suppose $\mathbf{u}_1 = -\mathbf{u}$. Then we set $\mathbf{v} = -\mathbf{w}_2$ and $\theta = 2\pi - \alpha$. On the other hand, if $\mathbf{u}_2 = -\mathbf{u}$, then we set $\mathbf{v} = \mathbf{w}_1$, and $\theta = \alpha = \pi$. Now suppose neither of $\mathbf{u}_1$ or $\mathbf{u}_2$ is opposite $\mathbf{u}$. Then setting $\mathbf{v} = \mathbf{w}_1$ will give the correct parametrization if either $\mathbf{w}_1 = -\mathbf{w}_2$, or if $\mathbf{w}_1 = \mathbf{w}_2$ and $|q_0 - q_1| < |q_0 - q_2|$. In the case that $\mathbf{w}_1 = \mathbf{w}_2$ and $|q_0 - q_1| > |q_0 - q_2|$, we set $\mathbf{v} = \mathbf{w}_2$. If $\mathbf{w}_1 = -\mathbf{w}_2$, or $\mathbf{w}_1 = \mathbf{w}_2$ and $|q_0 - q_1| > |q_0 - q_2|$, then we set $\theta = 2\pi - \alpha$. If $\mathbf{w}_1 = \mathbf{w}_2$ and $|q_0 - q_1| < |q_0 - q_2|$, then we set $\theta = \alpha$.
3. The reparametrization. In order to blend two circular arcs, C_1 and C_2, we will need to reparametrize the pieces which go from q_1 to q_2 in each case over the same interval, say $0 \le s \le 1$. First let β be defined so that $C_1(\beta) = q_1$ (computed in a similar way to α above), and let γ be defined so that $C_2(\gamma) = q_2$. The new parametrizations are then: $C_1((1-s)\beta + s\alpha)$ and $C_2(s\gamma)$, $0 \le s \le 1$.
4. Degree of smoothness at the join points

Splines as Optimal Curves

Here we review some of the properties of spline curves in $\mathbb{R}^n$. The most fundamental property which distinguishes splines in some sense as optimal or ideal interpolation curves is the minimization property of the complete cubic spline.

We state it here as follows: Let $P_0, \ldots, P_n$ be points in $\mathbb{R}^n$, and consider all possible interpolating curves $\gamma(t)$, $0 \le t \le 1$, such that for some points $0 = t_0 < t_1 < \cdots < t_n$, we have $\gamma(t_i) = P_i$, for $0 \le i \le n$, and $\gamma''(t)$ is continuous for all t, $0 \le t \le 1$ (ie. γ is C^2). For any such curve γ we will also define

$$K_2(\gamma) = \int_0^1 |\gamma''(t)|^2\, dt.$$

Then amongst all such curves γ, the one with the smallest value of $K_2(\gamma)$ is in fact the complete cubic spline.

We note that this is NOT the total curvature of the path $\gamma(t)$. In fact, it can be shown that the minimal value of the total curvature $K(\gamma)$ is not achieved by the complete cubic spline, or any other C^2 curve, but is approached as such curves approach the path of the piecewise linear interpolation.

How far are the interpolations we have considered from this ideal type of curve?

Optimization Methods

The methods of Barr et al., use cubic basis curves and optimization based on the minimization of the Euler-Lagrange Error Functional. They also use penalty functions which are designed to keep the curves close to S^3.