

A Compact Differential Formula for the First Derivative of a Unit Quaternion Curve

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Abstract

This paper presents a compact differential formula for the first derivative of a unit quaternion curve defined on $SO(3)$ or S^3 . The formula provides a convenient way to compute the angular velocity of a rotating 3D solid. We demonstrate the effectiveness of this formula by deriving the differential properties of various unit quaternion curves [4, 5, 6, 7, 8] at the curve end points.

Keywords: Quaternion, rotation, S^3 , $SO(3)$, exponential, logarithm, differential, Jacobian matrix

1 Introduction

Given a single 3D solid object, its orientation can be uniquely specified by an element $q \in SO(3)$, where $SO(3)$ is the rotation group of R^3 (see [2, 5, 8]). The rotational motion of a 3D solid object can be uniquely specified by a connected path in the space $SO(3)$; thus the motion control problem is: how to modify the shape and speed of a path $q(t) \in SO(3)$ ($0 \leq t \leq 1$). The rotation group $SO(3)$ is a projective space which is constructed from the unit 3-sphere S^3 by identifying each pair $[q, -q]$ of two antipodal points q and $-q \in S^3$ as a single point $q \in SO(3)$ (see [2]); the local geometry of $SO(3)$ is thus identical to that of S^3 . Since the differential properties are local properties, we compute the quaternion curve differentials in S^3 instead of $SO(3)$.

Shoemake [7] introduced the unit quaternions to the computer graphics community for the purpose of controlling 3D solid rotations. By generalizing line segments in R^3 to geodesic great circular arcs in S^3 , Shoemake [7] extended the de Casteljau algorithm to generate Bézier quaternion curves in S^3 . Given four unit quaternions $q_i \in S^3$ ($i = 1, 2, 3, 4$), a cubic Bézier quaternion curve $q(t) \in S^3$ ($0 \leq t \leq 1$) is determined by the four control points q_i 's; for this curve $q(t)$, Shoemake [7] claimed that

$$q'(0) = 3 \cdot \gamma'_{q_1, q_2}(0) \quad \text{and} \quad q'(1) = 3 \cdot \gamma'_{q_3, q_4}(1),$$

where $\gamma_{q_i, q_j}(t)$ ($0 \leq t \leq 1$) is the geodesic circular arc connecting q_i and q_j in S^3 . However, the differentiation itself was considered to be rather challenging and the details were not given in [7].

Shoemake [8] used the formula:

$$dq^\alpha = q^\alpha \ln(q) d\alpha + \alpha q^{\alpha-1} dq, \quad (1)$$

in deriving the quaternion curve differentials on S^3 for the purpose of extending Boehm quadrangle [1] to S^3 . This formula was cited from the great work of Sir Hamilton [3]; however, the formula only holds under the *complanarity* condition: $q \cdot dq = dq \cdot q$ (see [3], pp. 148 and 453). Unfortunately, Shoemake [8] misinterpreted the meaning of dq , which is the differential $q'(t)$, as the logarithm $\log q$ [9]. (More details on the geometric meaning of $\log q$ are explained in §2.) One can easily check the failure of the above formula in Equation (1) from a simple example: $q(t)^2 \in S^3$ for which $\alpha(t) \equiv 2$; that is, we have

$$\frac{d}{dt}q(t)^2 = \frac{d}{dt}q(t) \cdot q(t) = q'(t) \cdot q(t) + q(t) \cdot q'(t) \neq 2q(t) \cdot q'(t),$$

since the quaternion multiplication is not commutative in general.

In this paper, we present a compact differential formula for the first derivative of a unit quaternion curve. Using this formula, we can easily show that the claims of Shoemake [7, 8] on the quaternion curve differentials actually hold at the curve end points. It is also easy to prove that Hanotiaux and Peroche [4] do not generate a Hermite quaternion curve which interpolates two given boundary angular velocities exactly; using the differential formula presented in this paper, we provide a simple way to remedy this limitation. Furthermore, the quaternion calculus can be greatly simplified for computing the first derivative of a circular blending quaternion curve [5, 6]. However, the differential formula of this paper is only useful for the first derivative of a unit quaternion curve. For higher order derivatives such as the ones used in [5] to prove the C^k -continuity of a circular blending quaternion curve, we still need to use a more complex formula [5, 6].

The rest of this paper is organized as follows. In §2, mathematical preliminaries are given and the differential formula is derived for a unit quaternion curve. In §3, we apply the differential formula to prove the differential properties of various unit quaternion curves (see [4, 5, 6, 7, 8]). Finally, in §4, we conclude this paper.

2 Differential Formula

2.1 Logarithmic and Exponential Maps

Given a unit quaternion $q = (w, x, y, z) \in S^3$ (i.e., $w^2 + x^2 + y^2 + z^2 = 1$), let

$$\theta = \arccos w \in [0, \pi] \quad \text{and} \quad (a, b, c) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \in S^2.$$

Then we have

$$\begin{aligned} w &= \cos \theta, \\ \sin \theta &= \sqrt{1 - \cos^2 \theta} = \sqrt{1 - w^2} = \sqrt{x^2 + y^2 + z^2}, \quad \text{and} \\ (x, y, z) &= \sqrt{x^2 + y^2 + z^2} \cdot (a, b, c) = \sin \theta \cdot (a, b, c). \end{aligned} \quad (2)$$

Thus, the unit quaternion $q \in S^3$ can be represented by:

$$q = (\cos \theta, \sin \theta \cdot (a, b, c)),$$

where $0 \leq \theta \leq \pi$ and $(a, b, c) \in S^2$.

A unit quaternion $q = (\cos \theta, \sin \theta \cdot (a, b, c)) \in S^3$ maps each point $p = (x, y, z) \in R^3$ into a point $p' = (x', y', z') \in R^3$ which is given by the relation (see [2, 8]):

$$(0, x', y', z') = q \cdot (0, x, y, z) \cdot \bar{q},$$

where the \cdot operation is the quaternion multiplication and \bar{q} is the conjugate quaternion of q , i.e., $\bar{q} = (\cos \theta, -\sin \theta \cdot (a, b, c)) \in S^3$. One can easily check that the point p' is the same as the one which is obtained by rotating the point p about an axis (a, b, c) by an angle 2θ . Thus each unit quaternion $q \in S^3$ represents such a rotation; at the same time, it represents the orientation (of a 3D solid) which is obtained by rotating the solid from the standard orientation (which is represented by the identity quaternion $1 \equiv (1, 0, 0, 0)$) about an axis (a, b, c) by an angle 2θ . Note that the two antipodal points q and $-q$ of S^3 give the same rotated point $(x', y', z') \in R^3$; thus the two unit quaternions q and $-q \in S^3$ represent the same orientation. By identifying each pair of two antipodal points q and $-q \in S^3$ as a single orientation, the rotation group $SO(3)$ is obtained as a ~~quotient projective~~ space of S^3 .

Under the rotation of ~~about~~ an axis $(a, b, c) \in S^2$ by an angle 2θ , each intermediate orientation obtained is the same as the one which is obtained by rotating the 3D solid about the same axis $(a, b, c) \in S^2$ by an angle $2\theta t$ for some $t \in [0, 1]$. When we rotate a given 3D solid from the standard orientation, the trace of all the intermediate orientations can be represented by the geodesic circular arc $\gamma_{1,q}(t) \in S^3$, $0 \leq t \leq 1$, which connects the two unit quaternions: 1 and q ; that is,

$$\gamma_{1,q}(t) = (\cos \theta t, \sin \theta t \cdot (a, b, c)), \quad \text{for } 0 \leq t \leq 1.$$

The geodesic circular arc in S^3 or $SO(3)$ is also called as *slerp* in computer graphics (following its first usage in Shoemake [7]). The velocity for the curve $\gamma_{1,q}(t)$ is given by:

$$\begin{aligned} \gamma'_{1,q}(t) &= (-\theta \sin \theta t, \theta \cos \theta t \cdot (a, b, c)) \\ &= (0, \theta \cdot (a, b, c)) \cdot (\cos \theta t, \sin \theta t \cdot (a, b, c)) \\ &= (0, \theta \cdot (a, b, c)) \cdot \gamma_{1,q}(t), \end{aligned}$$

for $0 \leq t \leq 1$, where the \cdot operation between two 4D vectors means the quaternion multiplication. In this paper, the \cdot operation is interpreted as scalar multiplication, inner product, or quaternion multiplication depending on the context. The initial velocity is thus given by:

$$\gamma'_{1,q}(0) = (0, \theta \cdot (a, b, c)).$$

The mapping from each unit quaternion $q = (\cos \theta, \sin \theta \cdot (a, b, c))$ to the initial velocity $(0, \theta \cdot (a, b, c))$ gives a well-defined map from the 3-sphere S^3 (of unit quaternions) into the tangent space $T_1 S^3 \equiv R^3$. The tangent space $T_1 S^3$ is the space of all angular velocities from the standard orientation. The Talyor series expansion of the logarithmic function for unit quaternions realizes such a map (see [2]):

$$\log(\cos \theta, \sin \theta \cdot (a, b, c)) = (0, \theta \cdot (a, b, c)). \quad (3)$$

It is easier to prove the inverse relation:

$$\exp(0, \theta \cdot (a, b, c)) = (\cos \theta, \sin \theta \cdot (a, b, c))$$

by expanding the Taylor series of the exponential function for $(0, \theta \cdot (a, b, c))$; see [5, 6]. Thus the unit quaternion curve $\gamma_{1,q}(t)$ ($0 \leq t \leq 1$) can be represented by the log and exp maps:

$$\gamma_{1,q}(t) = (\cos \theta t, \sin \theta t \cdot (a, b, c))$$

$$\begin{aligned}
&= \exp(0, \theta t \cdot (a, b, c)) \\
&= \exp(t \cdot (0, \theta \cdot (a, b, c))) \\
&= \exp(t \cdot \log(\cos \theta, \sin \theta \cdot (a, b, c))) \\
&= \exp(t \cdot \log(q)).
\end{aligned}$$

Furthermore, the geodesic circular arc $\gamma_{q_1, q_2}(t) \in S^3$ ($0 \leq t \leq 1$) which connects two unit quaternions q_1 and $q_2 \in S^3$ is given by (also see [5, 6]):

$$\gamma_{q_1, q_2}(t) = \exp(t \cdot \log(q_2 \cdot q_1^{-1})) \cdot q_1, \quad \text{for } 0 \leq t \leq 1. \quad (4)$$

The curve differential is also given by:

$$\gamma'_{q_1, q_2}(t) = \gamma'_{1, q_2 \cdot q_1^{-1}}(t) \cdot q_1 = \log(q_2 \cdot q_1^{-1}) \cdot \gamma_{q_1, q_2}(t), \quad \text{for } 0 \leq t \leq 1.$$

Using the identities in Equation (2) and ignoring the first component of the right-hand side of Equation (3) which is always 0, we can define the logarithmic map $\log : S^3 \rightarrow R^3$ as follows (see Figure 1).

Definition 2.1 *The logarithmic map $\log : S^3 \rightarrow R^3$ is defined by:*

$$\log(w, x, y, z) = (\arccos w \cdot \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}) = (\frac{\arccos w}{\sqrt{1 - w^2}} \cdot (x, y, z)), \quad \text{for } (w, x, y, z) \in S^3.$$

Furthermore, given an initial velocity $v = (x, y, z) = \theta \cdot (a, b, c)$, by using the relation

$$\begin{aligned}
\theta &= \|v\| = \sqrt{x^2 + y^2 + z^2} \\
(a, b, c) &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \cdot (x, y, z) = \frac{1}{\|v\|} \cdot (x, y, z),
\end{aligned}$$

the exponential map \exp (which is the inverse map of \log) can be defined as follows.

Definition 2.2 *The exponential map $\exp : R^3 \rightarrow S^3 \subset R^4$ is defined by:*

$$\exp(x, y, z) = \begin{cases} (\cos \|v\|, \frac{\sin \|v\|}{\|v\|} \cdot (x, y, z)) & \text{if } v = (x, y, z) \neq (0, 0, 0) \\ (1, 0, 0, 0) & \text{if } v = (x, y, z) = (0, 0, 0) \end{cases}$$

2.2 Differential Formula for an Exponential Map

We are familiar with the differential of a one-variable real-valued function $f : R \rightarrow R$; that is, the first derivative $f'(x)$ of f at $x \in R$ is defined by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

where $h \in R$. To compute the differential of $\exp : R^3 \rightarrow R^4$, we need a more general definition of the differential for a multi-variate multi-valued map.

Definition 2.3 *A map $F : R^n \rightarrow R^m$ is differentiable at $p = (x_1, \dots, x_n) \in R^n$ if and only if there is a linear map $L : R^n \rightarrow R^m$ such that*

$$\lim_{h \rightarrow (0, \dots, 0)} \frac{\|F(p+h) - F(p) - L(h)\|}{\|h\|} = 0,$$

where $h \in R^n$. The linear map L is denoted by dF_p and called as the differential of F at p .

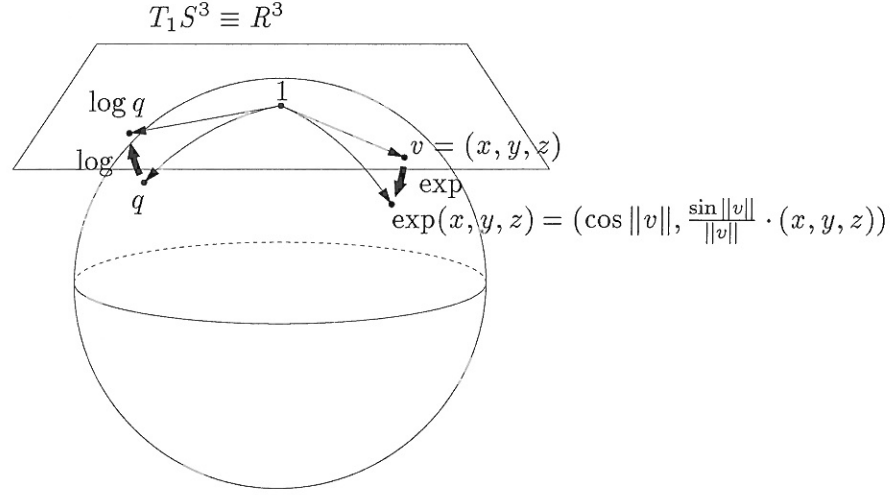


Figure 1: Log and Exp Maps.

The computation of dF_p with this definition is not easy since it requires to guess the linear map L and then to prove that the above limit actually goes to 0. When the coordinate functions of F are given by well-defined differentiable functions in a neighborhood of $p \in R^n$, there is a convenient way of computing the differential dF_p in terms of the partial derivatives of each of the component functions of F . This is given in Theorem 2.4 (see [10] for the proof).

Theorem 2.4 *For a differentiable map $F : R^n \rightarrow R^m$ which is defined by*

$$F(p) = F(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \in R^m,$$

for $p = (x_1, \dots, x_n) \in R^n$, the differential dF_p is given by the $m \times n$ Jacobian matrix:

$$dF_p = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \dots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}$$

In Definition 2.1, the component functions of the exponential map \exp are differentiable functions for $(x, y, z) \neq (0, 0, 0)$. Thus, by applying Theorem 2.4, we can easily obtain the differential formula $d\exp_{(x,y,z)}$ for $(x, y, z) \neq (0, 0, 0)$. (See Figure 2.)

Theorem 2.5 *For $v = (x, y, z) \neq (0, 0, 0)$, the exponential map $\exp : R^3 \rightarrow R^4$ has its differential $d\exp_{(x,y,z)}$ represented by a 4×3 Jacobian matrix:*

$$d\exp_{(x,y,z)} = \begin{bmatrix} -\frac{s}{\|v\|} \cdot x & -\frac{s}{\|v\|} \cdot y & -\frac{s}{\|v\|} \cdot z \\ \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot x^2 + \frac{s}{\|v\|} & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot xy & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot xz \\ \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot xy & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot y^2 + \frac{s}{\|v\|} & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot yz \\ \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot xz & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot yz & \left(\frac{c}{\|v\|^2} - \frac{s}{\|v\|^3} \right) \cdot z^2 + \frac{s}{\|v\|} \end{bmatrix},$$

where $c = \cos \|v\|$ and $s = \sin \|v\|$.

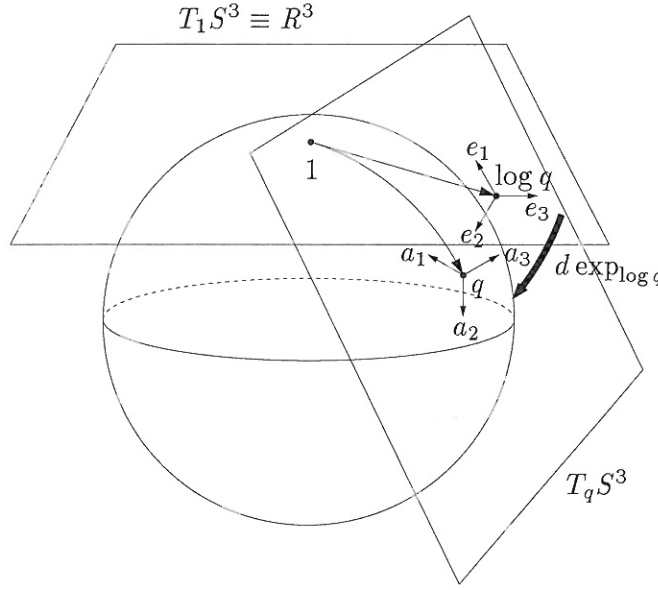


Figure 2: Differential of an Exponential Map.

In Figure 2, the differential $d\exp_{\log q}$ is illustrated as a mapping which maps each basis vector $e_i \in T_{\log q} R^3$ into the basis vector $a_i \in T_q S^3$ ($i = 1, 2, 3$). Note that the three column vectors of the Jacobian matrix $d\exp_{\log q}$ are identical with the three basis vectors $a_1, a_2, a_3 \in T_q S^3$.

The differentiability of the exponential map \exp at $v = (0, 0, 0)$ is not clear from the definition of \exp ; thus we prove the differentiability of \exp at $(0, 0, 0)$ and compute the differential $d\exp_{(0,0,0)}$ by using Definition 2.3. For this purpose, we first need to guess the linear transformation L ; the columns of L can be computed by taking the partial derivatives of \exp with respect to x, y, z .

Theorem 2.6 *For $v = (0, 0, 0)$, the differential $d\exp_{(0,0,0)}$ is given by:*

$$d\exp_{(0,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof: Let $h = (x, y, z) \in R^3$ be a 3D vector which approaches to the origin $(0, 0, 0) \in R^3$, and L be the above 4×3 matrix; then the following relation shows that $d\exp_{(0,0,0)} \equiv L$ by Definition 2.3.

$$\begin{aligned} & \lim_{h \rightarrow (0,0,0)} \frac{\|\exp(x, y, z) - \exp(0, 0, 0) - L(x, y, z)\|}{\|h\|} \\ = & \lim_{h \rightarrow (0,0,0)} \frac{\|(\cos \|h\|, \frac{\sin \|h\|}{\|h\|} \cdot (x, y, z)) - (1, 0, 0, 0) - (0, x, y, z)\|}{\|h\|} \\ = & \lim_{h \rightarrow (0,0,0)} \frac{\|(\cos \|h\| - 1, (\frac{\sin \|h\|}{\|h\|} - 1) \cdot (x, y, z))\|}{\|h\|} \\ = & \lim_{h \rightarrow (0,0,0)} \frac{\|(-\frac{1}{2} \cdot \|h\|^2 + o(\|h\|^4), (-\frac{1}{6} \cdot \|h\|^2 + o(\|h\|^4)) \cdot (x, y, z))\|}{\|h\|} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow (0,0,0)} \left\| \left(-\frac{1}{2} \cdot \|h\| + o(\|h\|^3), \left(-\frac{1}{6} \cdot \|h\| + o(\|h\|^3) \right) \cdot (x, y, z) \right) \right\| \\
&= \|(0, 0, 0, 0)\| \\
&= 0. \quad \blacksquare
\end{aligned}$$

2.3 Differential Formula for a Unit Quaternion Curve

The differential formula for a unit quaternion curve $q(t) \in S^3$ ($0 \leq t \leq 1$) is obtained by decomposing the quaternion curve $q(t)$ into two maps, \exp and $\log q$, which can be realized as maps between two Euclidean spaces and then by applying the following chain rule to the composite map (see [10] for the proof).

Theorem 2.7 (Chain Rule) *If $F : R^n \rightarrow R^m$ is differentiable at $p \in R^n$ and $G : R^m \rightarrow R^k$ is differentiable at $F(p) \in R^m$, then the composite map $G \circ F : R^n \rightarrow R^k$ is differentiable at $p \in R^n$. Furthermore, the differential of the composite map is given by: $d(G \circ F)_p = dG_{F(p)} \cdot dF_p$.*

For a unit quaternion curve $q(t) \in S^3$ ($0 \leq t \leq 1$), the curve can be represented by

$$q(t) = \exp(\log q(t)), \quad \text{for } 0 \leq t \leq 1.$$

Since we have $q : R \rightarrow S^3$ and $\log : S^3 \rightarrow R^3$, the composite map $p(t) = \log q(t)$ ($0 \leq t \leq 1$) defines a map $p : R \rightarrow R^3$; that is, we have

$$p(t) = \log q(t) = (x(t), y(t), z(t)) \in R^3, \quad \text{for } 0 \leq t \leq 1,$$

which defines a space curve in the 3D Euclidean space R^3 . Thus the quaternion curve $q : R \rightarrow S^3 \subset R^4$ can be decomposed into two maps, $p : R \rightarrow R^3$ and $\exp : R^3 \rightarrow S^3 \subset R^4$; the above chain rule can be applied to the composite map

$$q(t) = \exp(\log q(t)) = \exp(p(t)), \quad \text{for } 0 \leq t \leq 1.$$

Theorem 2.8 *For a unit quaternion curve $q(t) \in S^3$ and the space curve $p(t) = \log q(t) = (x(t), y(t), z(t)) \in R^3$ ($0 \leq t \leq 1$), the differential $q'(t)$ is given by:*

$$q'(t) = d\exp_{p(t)}(p'(t)) = d\exp_{(x(t), y(t), z(t))}(x'(t), y'(t), z'(t)).$$

Especially, for $t \in [0, 1]$ such that $p(t) = (x(t), y(t), z(t)) = (0, 0, 0)$, from Theorem 2.6, we have a simpler formula:

Corollary 2.9 *For a unit quaternion curve $q(t) \in S^3$ and the space curve $p(t) = \log q(t) = (x(t), y(t), z(t)) \in R^3$ ($0 \leq t \leq 1$), the differential $q'(t)$ at $t \in [0, 1]$ such that $p(t) = (0, 0, 0)$ is given by:*

$$q'(t) = d\exp_{(0,0,0)}(p'(t)) = p'(t) = (x'(t), y'(t), z'(t)).$$

3 Applications

In this section, we apply the differential formula derived in the previous section to prove and/or simplify some of the curve differential properties which have been claimed and/or used in the previous works [5, 6, 7, 8].

3.1 Blending Quaternion Curve

We first consider a quaternion curve $q(t) \in S^3$ which is obtained by blending two quaternions curves $q_1(t)$ and $q_2(t) \in S^3$ with a blending function $f(t) \in R$ ($0 \leq t \leq 1$). That is, for each $t \in [0, 1]$, the unit quaternion $q(t) \in S^3$ is obtained as the point which subdivides $\gamma_{q_1(t), q_2(t)}$ in the ratio of $f(t) : 1 - f(t)$, where $\gamma_{q_1(t), q_2(t)}$ is the geodesic circular arc which connects $q_1(t)$ and $q_2(t)$ on S^3 . The blending quaternion curve $q(t)$ is given by the following analytic formula (see Equation (4) and [5, 6]):

$$\begin{aligned} q(t) &= \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q_1(t) \\ &= \exp((1 - f(t)) \cdot \log(q_1(t) \cdot q_2(t)^{-1})) \cdot q_2(t), \quad \text{for } 0 \leq t \leq 1. \end{aligned}$$

The Bézier quaternion curve [7], the spherical quadrangle curve [8], and the circular blending quaternion curve [5, 6] all belong to this class of blending quaternion curves. Thus the differential formula derived in Theorem 3.1 and Corollary 3.2 can be directly applied to these quaternion curves.

Theorem 3.1 *Given two quaternion curves $q_1(t)$ and $q_2(t) \in S^3$, the blending quaternion curve $q(t) \in S^3$ of $q_1(t)$ and $q_2(t)$ with respect to a blending function $f(t) \in R$ ($0 \leq t \leq 1$) has its differential $q'(t)$ given by:*

$$\begin{aligned} q'(t) &= d \exp_{(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}))} \left(f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}) + f(t) \cdot \frac{d}{dt} \log(q_2(t) \cdot q_1(t)^{-1}) \right) \cdot q_1(t) \\ &\quad + \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q'_1(t). \end{aligned}$$

Proof:

$$\begin{aligned} q'(t) &= \left[\frac{d}{dt} \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \right] \cdot q_1(t) + \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q'_1(t) \\ &= d \exp_{(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}))} \left(\frac{d}{dt} (f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \right) \cdot q_1(t) \\ &\quad + \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q'_1(t) \\ &= d \exp_{(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}))} \left(f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}) + f(t) \cdot \frac{d}{dt} \log(q_2(t) \cdot q_1(t)^{-1}) \right) \cdot q_1(t) \\ &\quad + \exp(f(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q'_1(t). \quad \blacksquare \end{aligned}$$

Corollary 3.2 *For $t \in [0, 1]$ with $f(t) = 0$, the differential $q'(t)$ is given by:*

$$q'(t) = f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}) \cdot q_1(t) + q'_1(t).$$

Proof:

$$\begin{aligned}
q'(t) &= d \exp_{(0, \log(q_2(t) \cdot q_1(t)^{-1}))} \left(f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}) + 0 \cdot \frac{d}{dt} \log(q_2(t) \cdot q_1(t)^{-1}) \right) \cdot q_1(t) \\
&\quad + \exp(0 \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q'_1(t) \\
&= d \exp_{(0,0,0)}(f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1})) \cdot q_1(t) + \exp(0,0,0) \cdot q'_1(t) \\
&= f'(t) \cdot \log(q_2(t) \cdot q_1(t)^{-1}) \cdot q_1(t) + q'_1(t). \quad \blacksquare
\end{aligned}$$

3.2 B zier Quaternion Curve

Given four unit quaternions $q_i \in S^3$ ($i = 1, 2, 3, 4$), Shoemake [7] constructs a cubic B zier quaternion curve $q_{4,1}(t)$ ($0 \leq t \leq 1$) by extending the de Casteljau algorithm to S^3 , that is, by using geodesic circular arcs of S^3 instead of line segments of R^3 in the algorithm. Shoemake [7] claimed that

$$q'_{4,1}(0) = 3 \cdot \gamma'_{q_1, q_2}(0) \quad \text{and} \quad q'_{4,1}(1) = 3 \cdot \gamma'_{q_3, q_4}(1),$$

where γ_{q_i, q_j} is the geodesic circular arc which connects the two unit quaternions q_i and q_j in S^3 . However, no details are given in Shoemake [7]. Using the differential formula which is developed in Corollary 3.2, we can easily show this differential property of the B zier quaternion curve.

We consider the general case of B zier quaternion curve $q_{n,1}(t) \in S^3$ which is defined by n control points of unit quaternions $q_i \in S^3$ ($i = 1, \dots, n$). We show that

$$\begin{aligned}
q'_{n,1}(0) &= (n-1) \cdot \gamma'_{q_1, q_2}(0) = (n-1) \cdot \log(q_2 \cdot q_1^{-1}) \cdot q_1, \\
q'_{n,1}(1) &= (n-1) \cdot \gamma'_{q_{n-1}, q_n}(1) = (n-1) \cdot \log(q_n \cdot q_{n-1}^{-1}) \cdot q_n.
\end{aligned}$$

We prove this relation by an induction on n . First of all, each B zier quaternion curve $q_{m,i}(t)$ denotes the B zier quaternion curve which is constructed by m control points of unit quaternions $q_i, \dots, q_{i+m-1} \in S^3$. For $n = 2$, the quaternion curve $q_{2,1}(t)$ is given by the geodesic circular arc $\gamma_{q_1, q_2}(t)$ ($0 \leq t \leq 1$); that is,

$$q_{2,1}(t) = \gamma_{q_1, q_2}(t), \quad \text{for } 0 \leq t \leq 1.$$

It is clear that

$$q'_{2,1}(0) = \gamma'_{q_1, q_2}(0) \quad \text{and} \quad q'_{2,1}(1) = \gamma'_{q_1, q_2}(1).$$

The quaternion curve $q_{n,1}(t)$ is recursively defined by

$$q_{n,1}(t) = \exp(t \cdot \log(q_{n-1,2}(t) \cdot q_{n-1,1}(t)^{-1})) \cdot q_{n-1,1}(t),$$

for $0 \leq t \leq 1$. The blending function $f(t) = t$ has the condition: $f(0) = 0$ and $f'(0) = 1$; by Corollary 3.2 and the induction hypothesis, we have:

$$\begin{aligned}
q'_{n,1}(0) &= \log(q_{n-1,2}(0) \cdot q_{n-1,1}(0)^{-1}) \cdot q_{n-1,1}(0) + q'_{n-1,1}(0) \\
&= \log(q_2 \cdot q_1^{-1}) \cdot q_1 + (n-2) \cdot \log(q_2 \cdot q_1^{-1}) \cdot q_1 \\
&= (n-1) \cdot \log(q_2 \cdot q_1^{-1}) \cdot q_1 \\
&= (n-1) \cdot \gamma'_{q_1, q_2}(0).
\end{aligned}$$

The curve $q_{n,1}(t)$ can also be defined by

$$q_{n,1}(t) = \exp((1-t) \cdot \log(q_{n-1,1}(t) \cdot q_{n-1,2}(t)^{-1})) \cdot q_{n-1,2}(t), \quad \text{for } 0 \leq t \leq 1.$$

The blending function: $f(t) = 1 - t$, has the condition: $f(1) = 0$ and $f'(1) = -1$; by Corollary 3.2 and the induction hypothesis, we have:

$$\begin{aligned}
q'_{n,1}(1) &= -\log(q_{n-1,1}(1) \cdot q_{n-1,2}(1)^{-1}) \cdot q_{n-1,2}(1) + q'_{n-1,2}(1) \\
&= -\log(q_{n-1} \cdot q_n^{-1}) \cdot q_n + (n-2) \cdot \log(q_n \cdot q_{n-1}^{-1}) \cdot q_n \\
&= \log(q_n \cdot q_{n-1}^{-1}) \cdot q_n + (n-2) \cdot \log(q_n \cdot q_{n-1}^{-1}) \cdot q_n \\
&= (n-1) \cdot \log(q_n \cdot q_{n-1}^{-1}) \cdot q_n \\
&= (n-1) \cdot \gamma'_{q_{n-1}, q_n}(1).
\end{aligned}$$

3.3 Spherical Quadrangle Curve

Given four unit quaternions $q_i \in S^3$ ($i = 1, 2, 3, 4$), Shoemake [8] constructs a spherical quadrangle curve $q(t)$ ($0 \leq t \leq 1$) by extending the quadrangle curve of Boehm [1]; that is, the curve $q(t)$ is defined by

$$q(t) = \exp(2t(1-t) \cdot \log(\gamma_{q_2, q_3}(t) \cdot \gamma_{q_1, q_4}(t)^{-1})) \cdot \gamma_{q_1, q_4}(t), \text{ for } 0 \leq t \leq 1.$$

Since the blending function $f(t) = 2t(1-t)$ has the condition: $f(0) = f(1) = 0$, $f'(0) = 2$, and $f'(1) = -2$, we have:

$$\begin{aligned}
q'(0) &= 2\log(\gamma_{q_2, q_3}(0) \cdot \gamma_{q_1, q_4}(0)^{-1}) \cdot \gamma_{q_1, q_4}(0) + \gamma'_{q_1, q_4}(0) \\
&= 2\log(q_2 \cdot q_1^{-1}) \cdot q_1 + \log(q_4 \cdot q_1^{-1}) \cdot q_1, \\
q'(1) &= -2\log(\gamma_{q_2, q_3}(1) \cdot \gamma_{q_1, q_4}(1)^{-1}) \cdot \gamma_{q_1, q_4}(1) + \gamma'_{q_1, q_4}(1) \\
&= -2\log(q_3 \cdot q_4^{-1}) \cdot q_4 + \log(q_4 \cdot q_1^{-1}) \cdot q_4.
\end{aligned}$$

3.4 Circular Blending Curve

Given two unit quaternions q_1 and q_2 , and two circular quaternion curves $C_1(t)$ and $C_2(t) \in S^3$ ($0 \leq t \leq 1$) which interpolate the two unit quaternions q_1 and q_2 :

$$C_1(0) = q_1 = C_2(0) \quad \text{and} \quad C_1(1) = q_2 = C_2(1),$$

Kim and Nam [5, 6] showed that the blending quaternion curve $q(t) \in S^3$ which is defined by

$$q(t) = \exp(t \cdot \log(C_2(t) \cdot C_1(t)^{-1})) \cdot C_1(t), \text{ for } 0 \leq t \leq 1,$$

satisfies

$$q'(0) = C'_1(0) \quad \text{and} \quad q'(1) = C'_2(1).$$

Since the blending function $f(t) = t$ has the condition: $f(0) = 0$ and $f'(0) = 1$, we have

$$\begin{aligned}
q'(0) &= \log(C_2(0) \cdot C_1(0)^{-1}) \cdot C_1(0) + C'_1(0) \\
&= \log(q_1 \cdot q_1^{-1}) \cdot q_1 + C'_1(0) \\
&= \log(1) \cdot q_1 + C'_1(0) \\
&= 0 \cdot q_1 + C'_1(0) \\
&= C'_1(0).
\end{aligned}$$

The curve $q(t)$ can also be defined by

$$q(t) = \exp((1-t) \cdot \log(C_1(t) \cdot C_2(t)^{-1})) \cdot C_2(t).$$

Since the blending function $f(t) = 1-t$ has the condition: $f(1) = 0$ and $f'(1) = -1$, we have

$$\begin{aligned} q'(1) &= -\log(C_1(1) \cdot C_2(1)^{-1}) \cdot C_2(1) + C_2'(1) \\ &= -\log(q_2 \cdot q_2^{-1}) \cdot q_2 + C_2'(1) \\ &= -\log(1) \cdot q_2 + C_2'(1) \\ &= 0 \cdot q_2 + C_2'(1) \\ &= C_2'(1). \end{aligned}$$

3.5 The Quaternion Curve of Hanotiaux and Peroche

Hanotiaux and Peroche [4] present a construction method (based on the exponential and logarithmic maps) to generate a unit quaternion curve $q(t) \in S^3$ which interpolates a given sequence of unit quaternions $q_i \in S^3$ ($i = 1, \dots, n$). Each unit quaternion q_i is projected into $\log q_i \in T_1 S^3 \equiv R^3$, and the velocity v_i at $\log q_i$ is approximated by:

$$v_i = \frac{1}{2} \cdot \log(q_{i+1} \cdot q_{i-1}^{-1}) \in T_1 S^3 \equiv R^3, \quad (5)$$

where $T_q S^3$ denotes the 3-dimensional tangent space of S^3 at $q \in S^3$. For each pair of two consecutive unit quaternions q_i and $q_{i+1} \in S^3$, a Hermite curve $p_i(t) \in R^3$ ($0 \leq t \leq 1$) is constructed that interpolates the four boundary conditions:

$$p_i(0) = \log q_i, \quad p_i(1) = \log q_{i+1}, \quad p_i'(0) = v_i, \quad \text{and} \quad p_i'(1) = v_{i+1}.$$

The quaternion curve $q_i(t) \in S^3$ (which is a subsegment of $q(t)$ between q_i and q_{i+1}) is then constructed by mapping the Hermite curve $p_i(t) \in R^3$ back to S^3 under the exponential map:

$$q_i(t) = \exp(p_i(t)), \quad \text{for } 0 \leq t \leq 1. \quad (6)$$

It is easy to check that $q_i(t)$ interpolates q_i and q_{i+1} :

$$\begin{aligned} q_i(0) &= \exp(p_i(0)) = \exp(\log q_i) = q_i, \\ q_i(1) &= \exp(p_i(1)) = \exp(\log q_{i+1}) = q_{i+1}. \end{aligned}$$

Furthermore, two consecutive unit quaternion curve segments $q_i(t)$ and $q_{i+1}(t) \in S^3$ ($0 \leq t \leq 1$) have the same velocity at the common end point $q_i(1) = q_{i+1} = q_{i+1}(0)$:

$$q_i'(1) = d\exp_{p_i(1)}(p_i'(1)) = d\exp_{\log q_{i+1}}(v_{i+1}) = d\exp_{p_{i+1}(0)}(p_{i+1}'(0)) = q_{i+1}'(0).$$

Thus a C^1 -continuous unit quaternion curve $q(t) \in S^3$ is constructed that interpolates a given sequence of unit quaternions $q_i \in S^3$ ($i = 1, \dots, n$).

In the above construction, note that any other assignment of v_i 's would also generate a C^1 -continuous unit quaternion curve $q(t) \in S^3$. Then, the question is: what is the most reasonable assignment of default values for v_i 's? Following the philosophy of Catmull-Rom spline tangents, we may claim that a reasonable choice for the tangential velocity $w_i \in T_{q_i} S^3$ of the curve $q(t) \in S^3$ at each q_i is:

$$w_i = \frac{1}{2} \cdot \log(q_{i+1} \cdot q_{i-1}^{-1}) \cdot q_i = v_i \cdot q_i \in T_{q_i} S^3,$$

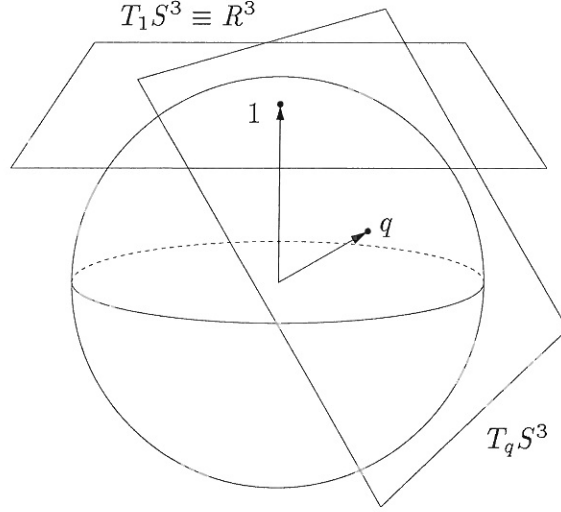


Figure 3: Tangent Spaces.

where $v_i \in T_1 S^3 \equiv R^3$ is the tangential velocity given in Equation (5). Since the quaternion multiplication gives a Lie group structure on S^3 , the transformation (which is called as the *right translation* by q_i):

$$\begin{aligned} R_{q_i} : S^3 &\longrightarrow S^3 \\ q &\longmapsto q \cdot q_i \end{aligned}$$

induces a linear transformation:

$$\begin{aligned} d(R_{q_i})_1 : T_1 S^3 &\longrightarrow T_{q_i} S^3. \\ v &\longmapsto v \cdot q_i \end{aligned}$$

The differential $d(R_{q_i})_1$ is an isomorphism with its inverse $d(R_{q_i^{-1}})_{q_i} : T_{q_i} S^3 \longrightarrow T_1 S^3$. Hanotiaux and Peroche [4] transformed the tangential velocity $w_i \in T_{q_i} S^3$ into the corresponding canonical tangential velocity $v_i = w_i \cdot q_i^{-1} \in T_1 S^3 \equiv R^3$ by doing a right translation by q_i^{-1} in the Lie group S^3 . This is necessary for the construction of a Hermite interpolation curve $p_i(t) \in R^3$ ($0 \leq t \leq 1$) in $T_1 S^3 \equiv R^3$; the 4D vector w_i can not be used directly for the construction of a Hermite curve in R^3 . Thus a 3D vector v_i is used instead for the Hermite interpolation. However, the choice of v_i in this way seems not reasonable.

The quaternion curve $q_i(t)$ constructed by Equation (6) does not interpolate the tangential velocities w_i and w_{i+1} at the curve end points (i.e., $q'_i(0) \neq w_i$ and $q'_i(1) \neq w_{i+1}$). This is because the velocity $v_i \in R^3$ does not map into the tangential velocity $w_i = v_i \cdot q_i \in T_{q_i} S^3$ under the differential $d \exp_{\log q_i}$:

$$q'_i(0) = d \exp_{p_i(0)}(p'_i(0)) = d \exp_{\log q_i}(v_i) \neq v_i \cdot q_i.$$

This fact shows that Hanotiaux and Peroche [4] do not generate an exact Hermite interpolation quaternion curve $q(t) \in S^3$ ($0 \leq t \leq 1$) which interpolates two boundary unit quaternions q_1 and $q_2 \in S^3$ and two boundary angular velocities $w_i \in T_{q_i} S^3$ ($i = 1, 2$). The algorithm of [4] guesses

v_i as $w_i \cdot q_i^{-1} \in R^3$; however, the right solution for $v_i \in R^3$ should be the one which satisfies the relation:

$$d\exp_{\log q_i}(v_i) = w_i, \quad \text{for } i = 1, 2.$$

Let A be the 4×3 Jacobian matrix of $d\exp_{\log q_i}$, and let a_1, a_2, a_3 be the three column vectors of A . The three 4D vectors a_1, a_2, a_3 span the tangent space T_{q_i} of S^3 (see Figure 2). Furthermore, since the angular velocity $w_i \in T_{q_i}S^3$ is a 4D vector which is tangent to S^3 , w_i can be uniquely represented by a linear combination of a_1, a_2, a_3 . The coefficients for the linear combination can be computed as the coordinates of a 3D vector $v_i \in R^3$ as follows. Let

$$Av_i = w_i, \quad \text{for } i = 1, 2,$$

then we have

$$A^t Av_i = A^t w_i \quad \text{and} \quad v_i = (A^t A)^{-1} A^t w_i.$$

Since the 4×3 matrix A has rank 3, the composite 3×3 matrix $A^t A$ has rank 3 (see [11], pp. 156–157); thus it is invertible and there is a unique solution v_i for the Hermite interpolation problem.

4 Conclusion

We have derived a compact differential formula for the first derivative of a unit quaternion curve in $SO(3)$ or S^3 . The effectiveness of this formula is demonstrated in deriving the differential properties of various unit quaternion curves [4, 5, 6, 7, 8]. However, the differential formula of this paper is only useful for the first derivative of a unit quaternion curve; for higher order derivatives, we still need to use a more complex formula (see [5]).

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