

**Graduate Texts in Mathematics**  
*Readings in Mathematics*

Ebbinghaus/Hermes/Hirzebruch/Koecher/Mainzer/Neukirch/Prestel/Remmert: *Numbers*  
Remmert: *Theory of Complex Functions*

**Undergraduate Texts in Mathematics**  
*Readings in Mathematics*

Samuel: *Projective Geometry*

H.-D. Ebbinghaus    H. Hermes  
F. Hirzebruch    M. Koecher    K. Mainzer  
J. Neukirch    A. Prestel    R. Remmert

# Numbers

With an Introduction by K. Lamotke  
Translated by H.L.S. Orde  
Edited by J.H. Ewing

With 24 Illustrations



Springer-Verlag  
New York Berlin Heidelberg  
London Paris Tokyo Hong Kong

Heinz-Dieter Ebbinghaus      Max Koecher (1924–1990)  
 Hans Hermes      Reinhold Remmert  
 Mathematisches Institut      Mathematisches Institut Universität  
 Universität Freiburg      Münster  
 Albertstraße 23b, D-7800      Einsteinstraße 62, D-4400 Münster,  
 Freiburg, FRG      FRG

Friedrich Hirzebruch      Klaus Mainzer  
 Max-Planck-Institut für      Lehrstuhl für Philosophie und  
 Mathematik      Wissenschaftstheorie  
 Gottfried-Claren-Straße 26      Universität Augsburg  
 D-5300 Bonn 3, FRG      Universitätsstraße 10  
    D-8900 Augsburg, FRG

Klaus Lamotte (*Editor of*      H.L.S. Orde (*Translator*)  
*German Edition*)      Bressenden  
 Mathematisches Institut      Biddenden near Ashford  
 der Universität zu Köln      Kent TN27 8DU, UK  
 Weyertal 86–90, D-5000      Weyertal 86–90, D-5000  
 Köln, FRG      Köln, FRG

*Editorial Board*  
 J.H. Ewing      F.W. Gehring  
 Department of Mathematics      Department of Mathematics  
 Indiana University      University of Michigan  
 Bloomington, IN 47405, USA      Ann Arbor, MI 48019, USA

John H. Ewing (*Editor of*      P.R. Halmos  
*English Edition*)      Department of Mathematics  
 Department of Mathematics      Santa Clara University  
 Indiana University      Santa Clara, CA 95053, USA  
 Bloomington, IN 47405, USA

#### Mathematics Subject Classification (1980): 00A05

Library of Congress Cataloging-in-Publication Data  
 Zahlen, Grundwissen Mathematik I. English

Numbers / Heinz-Dieter Ebbinghaus . . . [et al.] ; with an  
 introduction by Klaus Lamotte ; translated by H.L.S. Orde ; edited  
 by John H. Ewing.

p. cm.—(Readings in mathematics)  
 Includes bibliographical references.  
 ISBN 0-387-97202-1

I. Number theory. I. Ebbinghaus, Heinz-Dieter. II. Ewing,  
 John H. III. Series: Undergraduate texts in mathematics. Readings  
 in mathematics.  
 QA241.Z3413 1990  
 512'.7—dc20

89-48588

This book is a translation of the second edition of *Zahlen*, Grundwissen Mathematik I, Springer-Verlag, 1988.

© 1990 Springer-Verlag New York Inc.  
 All rights reserved. This work may not be translated or copied in whole or in part without the written permission  
 of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for  
 brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information  
 storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known  
 or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are  
 not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and  
 Merchandise Marks Act, may accordingly be used freely by anyone.

Camera-ready copy prepared using LaTeX.  
 Printed and bound by R.R. Donnelley & Sons, Harrisonburg, Virginia.  
 Printed in the United States of America.

9 8 7 6 5 4 3 2 1      Printed on acid-free paper.

ISBN 0-387-97202-1 Springer-Verlag New York Berlin Heidelberg  
 ISBN 3-540-97202-1 Springer-Verlag Berlin Heidelberg New York

## Preface to the English Edition

A book about numbers sounds rather dull. This one is not. Instead it is a  
 lively story about one thread of mathematics—the concept of “number”—  
 told by eight authors and organized into a historical narrative that leads  
 the reader from ancient Egypt to the late twentieth century. It is a story  
 that begins with some of the simplest ideas of mathematics and ends with  
 some of the most complex. It is a story that mathematicians, both amateur  
 and professional, ought to know.

Why write about numbers? Mathematicians have always found it diffi-  
 cult to develop broad perspective about their subject. While we each view  
 our specialty as having roots in the past, and sometimes having connec-  
 tions to other specialties in the present, we seldom see the panorama of  
 mathematical development over thousands of years. *Numbers* attempts to  
 give that broad perspective, from hieroglyphs to  $K$ -theory, from Dedekind  
 cuts to nonstandard analysis. Who first used the standard notation for  
 $\pi$  (and who made it standard)? Who were the “quaternionists” (and can  
 their zeal for quaternions tell us anything about the recent controversy  
 concerning Chaos)? What happened to the endless supply of “hypercom-  
 plex numbers” or to quaternionic function theory? How can the study of  
 maps from projective space to itself give information about algebras? How  
 did mathematicians resurrect the “ghosts of departed quantities” by rein-  
 troducing infinitesimals after 200 years? How can games be numbers and  
 numbers be games? This is mathematical culture, but it’s not the sort of  
 culture one finds in scholarly tomes; it’s lively culture, meant to entertain  
 as well as to inform.

This is not a book for the faint-hearted, however. While it starts with  
 material that every undergraduate could (and should) learn, the reader is  
 progressively challenged as the chapters progress into the twentieth century.  
 The chapters often tell about people and events, but they primarily tell  
 about mathematics. Undergraduates can certainly read large parts of this  
 book, but mastering the material in late chapters requires work, even for  
 mature mathematicians. This is a book that can be read on several levels,  
 by amateurs and professionals alike.

The German edition of this book, *Zahlen*, has been quite successful.  
 There was a temptation to abbreviate the English language translation  
 by making it less complete and more compact. We have instead tried to  
 produce a faithful translation of the entire original, which can serve as a  
 scholarly reference as well as casual reading. For this reason, quotations

are included along with translations and references to source material in foreign languages are included along with additional references (usually more recent) in English.

Translations seldom come into the world without some labor pains. Authors and translators never agree completely, especially when there are eight authors and one translator, all of whom speak both languages. My job was to act as referee in questions of language and style, and I did so in a way that likely made neither side happy. I apologize to all.

Finally, I would like to thank my colleague, Max Zorn, for his helpful advice about terminology, especially his insistence on the word "octonions" rather than "octaves."

March 1990

John Ewing

## Preface to Second Edition

The welcome which has been given to this book on numbers has pleasantly surprised the authors and the editor. The scepticism which some of us had felt about its concept has been dispelled by the reactions of students, colleagues and reviewers. We are therefore very glad to bring out a second edition—much sooner than had been expected. We have willingly taken up the suggestion of readers to include an additional chapter by J. NEUKIRCH on  $p$ -adic numbers. The chapter containing the theorems of FROBENIUS and HOPF has been enlarged to include the GELFAND–MAZUR theorem. We have also carefully revised all the other chapters and made some improvements in many places. In doing so we have been able to take account of many helpful comments made by readers for which we take this opportunity of thanking them. P. ULRICH of Münster who had already prepared the name and subject indexes for the first edition has again helped us with the preparation of the second edition and deserves our thanks.

Oberwolfach, March 1988

Authors and Publisher

## Preface to First Edition

The *basic mathematical knowledge* acquired by every mathematician in the course of his studies develops into a unified whole only through an awareness of the multiplicity of relationships between the individual mathematical theories. Interrelationships between the different mathematical disciplines often reveal themselves by studying historical development. One of the main underlying aims of this series is to make the reader aware that mathematics does not consist of isolated theories, developed side by side, but should be looked upon as an organic whole.

The present book on numbers represents a departure from the other volumes of the series inasmuch as seven authors and an editor have together contributed thirteen chapters. In conversations with one another the authors agreed on their contributions, and the editor endeavored to bring them into harmony by reading the contributions with a critical eye and holding subsequent discussions with the authors. The other volumes of the series can be studied independently of this one.

While it is impossible to name here all those who have helped us by their comments, we should nevertheless like to mention particularly Herr Gericke (of Freiburg) who helped us on many occasions to present the historical development in its true perspective.

K. Peters (at that time with Springer-Verlag) played a vital part in arranging the first meeting between the publisher and the authors. The meetings were made possible by the financial support of the Volkswagen Foundation and Springer-Verlag, as well as by the hospitality of the Mathematical Research Institute in Oberwolfach.

To all of these we extend our gratitude.

Oberwolfach, July 1983

Authors and Editor

# Contents

*Preface to the English Edition*

v

*Preface to Second Edition*

vii

*Preface to First Edition*

ix

*Introduction*, K. Lamotte

1

**Part A. From the Natural Numbers, to the Complex Numbers, to the  $p$ -adics**

7

*Chapter 1. Natural Numbers, Integers, and Rational Numbers.*  
K. Mainzer

9

§1. Historical

9

1. Egyptians and Babylonians. 2. Greece. 3. Indo-Arabic  
Arithmetical Practice. 4. Modern Times

§2. Natural Numbers

14

1. Definition of the Natural Numbers. 2. The Recursion  
Theorem and the Uniqueness of  $\mathbb{N}$ . 3. Addition, Multiplication  
and Ordering of the Natural Numbers. 4. PEANO's Axioms

§3. The Integers

19

1. The Additive Group  $\mathbb{Z}$ . 2. The Integral Domain  $\mathbb{Z}$ .

§4. The Rational Numbers

22

1. Historical. 2. The Field  $\mathbb{Q}$ . 3. The Ordering of  $\mathbb{Q}$

References

23

*Chapter 2. Real Numbers.* K. Mainzer

27

§1. Historical

27

1. HIPPOCRATES and the Pentagon. 2. EUDOXOS and the  
Theory of Proportion. 3. Irrational Numbers in Modern  
Mathematics. 4. The Formulation of More Precise Definitions  
in the Nineteenth Century

§2. DEDEKIND Cuts

36

1. The Set  $\mathbb{R}$  of Cuts. 2. The Order Relation in  $\mathbb{R}$ .

§3.	3. Addition in $\mathbb{R}$ . 4. Multiplication in $\mathbb{R}$ Fundamental Sequences	39
§4.	1. Historical Remarks. 2. CAUCHY's Criterion for Convergence. 3. The Ring of Fundamental Sequences. 4. The Residue Class Field $F/N$ of Fundamental Sequences. Modulo the Null Sequence. 5. The Completely Ordered Residue Class Field $F/N$	43
§5.	1. Nesting of Intervals Historical Remarks. 2. Nested Intervals and Completeness Axiomatic Definition of Real Numbers	46
§6.	1. The Natural Numbers, the Integers, and the Rational Numbers in the Real Number Field. 2. Completeness Theorem. 3. Existence and Uniqueness of the Real Numbers	51
References		51
<i>Chapter 3. Complex Numbers.</i> R. Remmert		55
§1.	Genesis of the Complex Numbers 1. CARDANO (1501-1576). 2. BOMBELLI (1526-1572). 3. DESCARTES (1596-1650), NEWTON (1643-1727) and LEIBNIZ (1646-1716). 4. EULER (1707-1783). 5. WALLIS (1616-1703), WESSEL (1745-1818) and ARGAND (1768-1822). 6. GAUSS (1777-1855). 7. CAUCHY (1789-1857). 8. HAMILTON (1805-1865). 9. Later Developments	56
§2.	The Field $\mathbb{C}$ 1. Definition by Pairs of Real Numbers. 2. The Imaginary Unit $i$ . 3. Geometric Representation. 4. Impossibility of Ordering the Field $\mathbb{C}$ . 5. Representation by Means of $2 \times 2$ Real Matrices	65
§3.	Algebraic Properties of the Field $\mathbb{C}$ 1. The Conjugation $\mathbb{C} \rightarrow \mathbb{C}$ , $z \mapsto \bar{z}$ . 2. The Field Automorphisms of $\mathbb{C}$ . 3. The Natural Scalar Product $\operatorname{Re}(w\bar{z})$ and Euclidean Length $ z $ . 4. Product Rule and the "Two Squares" Theorem. 5. Quadratic Roots and Quadratic Equations. 6. Square Roots and $n$ th Roots Geometric Properties of the Field $\mathbb{C}$	71
§4.	1. The Identity $(w, z)^2 + (iw, z)^2 =  w ^2 z ^2$ . 2. Cosine Theorem and the Triangle Inequality. 3. Numbers on Straight Lines and Circles. Cross-Ratio. 4. Cyclic Quadrilaterals and Cross- Ratio. 5. PTOLEMY's Theorem. 6. WALLACE's Line.	78
§5.	The Groups $O(\mathbb{C})$ and $SO(2)$ 1. Distance Preserving Mappings of $\mathbb{C}$ . 2. The Group $O(\mathbb{C})$ . 3. The Group $SO(2)$ and the Isomorphism $S^1 \rightarrow SO(2)$ .	85

§6.	4. Rational Parametrization of Properly Orthogonal $2 \times 2$ Matrices. Polar Coordinates and $n$ th Roots	89
§7.	1. Polar Coordinates. 2. Multiplication of Complex Numbers in Polar Coordinates. 3. DE MOIVRE's Formula. 4. Roots in Unity.	89
<i>Chapter 4. The Fundamental Theorem of Algebra.</i> R. Remmert		97
§1.	On the History of the Fundamental Theorem 1. GIRARD (1595-1632) and DESCARTES (1596-1650). 2. LEIBNIZ (1646-1716). 3. EULER (1707-1783). 4. D'ALEMBERT (1717-1783). 5. LAGRANGE (1736-1813) and LAPLACE (1749-1827). 6. GAUSS's Critique. 7. GAUSS's Four Proofs. 8. ARGAND (1768-1822) and CAUCHY (1798-1857). 9. The Fundamental Theorem of Algebra: Then and Now. 10. Brief Biographical Notes on Carl Friedrich GAUSS	98
§2.	Proof of the Fundamental Theorem Based on ARGAND 1. CAUCHY's Minimum Theorem. 2. Proof of the Fundamental Theorem. 3. Proof of ARGAND's Inequality. 4. Variant of the Proof. 5. Constructive Proofs of the Fundamental Theorem.	111
§3.	Application of the Fundamental Theorem 1. Factorization Lemma. 2. Factorization of Complex Polynomials. 3. Factorization of Real Polynomials. 4. Existence of Eigenvalues. 5. Prime Polynomials in $\mathbb{C}[Z]$ and $\mathbb{R}[X]$ . 6. Uniqueness of $\mathbb{C}$ . 7. The Prospects for "Hypercomplex Numbers."	115
	Appendix. Proof of the Fundamental Theorem, after LAPLACE 1. Results Used. 2. Proof. 3. Historical Note	120
<i>Chapter 5. What is <math>\pi</math>? R. Remmert</i>		123
§1.	On the History of $\pi$ 1. Definition by Measuring a Circle. 2. Practical Approxi- mations. 3. Systematic Approximation. 4. Analytical Formulae. 5. BALZER's Definition. 6. LANDAU and His Contemporary Critics	124
§2.	The Exponential Homomorphism $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$ 1. The Addition Theorem. 2. Elementary Consequences. 3. Epimorphism Theorem. 4. The Kernel of the Exponential Homomorphism. Definition of $\pi$ . Appendix. Elementary Proof of Lemma 3.	131
§3.	Classical Characterizations of $\pi$ 1. Definitions of $\cos z$ and $\sin z$ . 2. Addition Theorem.	137

3. The Number $\pi$ and the Zeros of $\cos z$ and $\sin z$ . 4. The Number $\pi$ and the Periods of $\exp z$ , $\cos z$ and $\sin z$ . 5. The Inequality $\sin y > 0$ for $0 < y < \pi$ and the Equation $e^{i\frac{\pi}{2}} = i$ . 6. The Polar Coordinate Epimorphism $p: \mathbb{R} \rightarrow S^1$ . 7. The Number $\pi$ and the Circumference and Area of a Circle.	142
§4. Classical Formulae for $\pi$	
1. LEIBNIZ'S Series for $\pi$ . 2. VIETA'S Product Formula for $\pi$ . 3. EULER'S Product for the Sine and WALLIS'S Product for $\pi$ . 4. EULER'S Series for $\pi^2, \pi^4, \dots$ 5. The WEIERSTRASS Definition of $\pi$ . 6. The Irrationality of $\pi$ and Its Continued Fraction Expansion. 7. Transcendence of $\pi$ .	

## Chapter 6. The $p$ -Adic Numbers. J. Neukirch

§1. Numbers as Functions	155
§2. The Arithmetic Significance of the $p$ -Adic Numbers	162
§3. The Analytical Nature of $p$ -Adic Numbers	166
§4. The $p$ -Adic Numbers	173
References	177

## Part B. Real Division Algebras

### Introduction, M. Koecher, R. Remmert

181

### Repertory. Basic Concepts from the Theory of Algebras,

M. Koecher, R. Remmert

183

1. Real Algebras. 2. Examples of Real Algebras. 3. Subalgebras and Algebra Homomorphisms. 4. Determination of All One-Dimensional Algebras. 5. Division Algebras. 6. Construction of Algebras by Means of Bases

### Chapter 7. Hamilton's Quaternions. M. Koecher, R. Remmert

189

#### Introduction

##### §1. The Quaternion Algebra $\mathbb{H}$

189

1. The Algebra  $\mathbb{H}$  of the Quaternions. 2. The Matrix Algebra  $\mathcal{H}$  and the Isomorphism  $F: \mathbb{H} \rightarrow \mathcal{H}$ . 3. The Imaginary Space of  $\mathbb{H}$ . 4. Quaternion Product, Vector Product and Scalar Product. 5. Noncommutativity of  $\mathbb{H}$ . The Center. 6. The Endomorphisms of the  $\mathbb{R}$ -Vector Space  $\mathbb{H}$ . 7. Quaternion Multiplication and Vector Analysis. 8. The Fundamental Theorem of Algebra for Quaternions.

##### §2. The Algebra $\mathbb{H}$ as a Euclidean Vector Space

194

##### §2. Conjugation and the Linear Form $\text{Re}$ . 2. Properties of

206

the Scalar Product. 3. The "Four Squares Theorem". 4. Preservation of Length, and of the Conjugacy Relation Under Automorphisms. 5. The Group $S^3$ of Quaternions of Length 1. 6. The Special Unitary Group $SU(2)$ and the Isomorphism $S^3 \rightarrow SU(2)$ .	213
§3. The Orthogonal Groups $O(3)$ , $O(4)$ and Quaternions	
1. Orthogonal Groups. 2. The Group $O(\mathbb{H})$ . CAYLEY'S Theorem. 3. The Group $O(\text{Im } \mathbb{H})$ . HAMILTON'S Theorem. 4. The Epimorphisms $S^3 \rightarrow SO(3)$ and $S^3 \times S^3 \rightarrow SO(4)$ . 5. Axis of Rotation and Angle of Rotation. 6. EULER'S Parametric Representation of $SO(3)$ .	

## Chapter 8. The Isomorphism Theorems of FROBENIUS, HOPF and GELFAND-MAZUR. M. Koecher, R. Remmert

221

#### Introduction

##### §1. Hamiltonian Triples in Alternative Algebras

221

1. The Purely Imaginary Elements of an Algebra.

##### §2. Hamiltonian Triple. 3. Existence of Hamiltonian Triples in Alternative Algebras. 4. Alternative Algebras.

223

##### §2. FROBENIUS'S Theorem

227

1. FROBENIUS'S Lemma. 2. Examples of Quadratic Algebras. 3. Quaternions Lemma. 4. Theorem of FROBENIUS (1877)

##### §3. HOPF'S Theorem

230

1. Topologization of Real Algebras. 2. The Quadratic Mapping  $\mathcal{A} \rightarrow \mathcal{A}$ ,  $x \mapsto x^2$ . HOPF'S Lemma. 3. HOPF'S Theorem.

4. The Original Proof by HOPF. 5. Description of All 2-Dimensional Algebras with Unit Element

##### §4. The GELFAND-MAZUR Theorem

238

1. BANACH Algebras. 2. The Binomial Series. 3. Local Inversion Theorem. 4. The Multiplicative Group  $\mathcal{A}^\times$ . 5. The GELFAND-MAZUR Theorem. 6. Structure of Normed Associative Division Algebras. 7. The Spectrum. 8. Historical Remarks on the GELFAND-MAZUR Theorem. 9. Further Developments

1. BANACH Algebras. 2. The Binomial Series. 3. Local Inversion Theorem. 4. The Multiplicative Group  $\mathcal{A}^\times$ . 5. The GELFAND-MAZUR Theorem. 6. Structure of Normed Associative Division Algebras. 7. The Spectrum. 8. Historical Remarks on the GELFAND-MAZUR Theorem. 9. Further Developments

## Chapter 9. CAYLEY Numbers or Alternative Division Algebras. M. Koecher, R. Remmert

249

##### §1. Alternative Quadratic Algebras

250

1. Quadratic Algebras. 2. Theorem on the Bilinear Form.

3. Theorem on the Conjugation Mapping. 4. The Triple Product Identity. 5. The Euclidean Vector Space  $\mathcal{A}$  and the Orthogonal Group  $O(\mathcal{A})$

##### §2. Existence and Properties of Octonions

256

1. Construction of the Quadratic Algebra  $\mathbb{O}$  of Octonions.



2. The Imaginary Space, Linear Form, Bilinear Form, and Conjugation of $\mathbb{O}$ . 3. $\mathbb{O}$ as an Alternative Division Algebra. 4. The "Eight-Squares" Theorem. 5. The Equation $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ . 6. Multiplication Table for $\mathbb{O}$	
§3. Uniqueness of the CAYLEY Algebra	261
1. Duplication Theorem. 2. Uniqueness of the CAYLEY Algebra (ZORN 1933). 3. Description of $\mathbb{O}$ by ZORN's Vector Matrices	

*Chapter 10. Composition Algebras. HURWITZ's Theorem—Vector-Product Algebras.* M. Koecher, R. Remmert

§1. Composition Algebras	267
1. Historical Remarks on the Theory of Composition.	
2. Examples. 3. Composition Algebras with Unit Element.	
4. Structure Theorem for Composition Algebras with Unit Element	
§2. Mutation of Composition Algebras	272
1. Mutation of Algebras. 2. Mutation Theorem for Finite-Dimensional Composition Algebras. 3. HURWITZ's Theorem (1898)	
§3. Vector-Product Algebras	275
1. The Concept of a Vector-Product Algebra. 2. Construction of Vector-Product Algebras. 3. Specification of all Vector-Product Algebras. 4. MAL'CEV-Algebras. 5. Historical Remarks	

*Chapter 11. Division Algebras and Topology.* F. Hirzebruch

§1. The Dimension of a Division Algebra Is a Power of 2	281
1. Odd Mappings and HOPF's Theorem. 2. Homology and Cohomology with Coefficients in $F_2$ . 3. Proof of HOPF's Theorem. 4. Historical Remarks on Homology and Cohomology Theory. 5. STURFEL's Characteristic Homology Classes	
§2. The Dimension of a Division Algebra Is 1, 2, 4 or 8	290
1. The mod 2 Invariants $\alpha(f)$ . 2. Parallelizability of Spheres and Division Algebras. 3. Vector Bundles. 4. WHITNEY's Characteristic Cohomology Classes. 5. The Ring of Vector Bundles. 6. Bott Periodicity. 7. Characteristic Classes of Direct Sums and Tensor Products. 8. End of the Proof. 9. Historical Remarks	
§3. Additional Remarks	299
1. Definition of the HOPF Invariant. 2. The HOPF Construction. 3. ADAM's Theorem on the HOPF Invariants. 4. Summary. 5. ADAM's Theorem About Vector Fields on Spheres	

References	301
------------	-----

**Part C. Infinitesimals, Games, and Sets**

*Chapter 12. Nonstandard Analysis.* A. Prestel

§1. Introduction	305
§2. The Nonstandard Number Domain ${}^*\mathbb{R}$	309
1. Construction of ${}^*\mathbb{R}$ . 2. Properties of ${}^*\mathbb{R}$	
§3. Features Common to $\mathbb{R}$ and ${}^*\mathbb{R}$	316
§4. Differential and Integral Calculus	321
1. Differentiation. 2. Integration	
Epilogue	326
References	327

*Chapter 13. Numbers and Games.* H. Hermes

§1. Introduction	329
1. The Traditional Construction of the Real Numbers.	
2. The CONWAY Method. 3. Synopsis	
§2. CONWAY Games	331
1. Discussion of the DEDKIND Postulates. 2. CONWAY's Modification of the DEDKIND Postulates. 3. CONWAY Games	
§3. Games	334
1. The Concept of a Game. 2. Examples of Games. 3. An Induction Principle for Games	
§4. On the Theory of Games	336
1. Winning Strategies. 2. Positive and Negative Games.	
§5. A Classification of Games	339
3. A Partially Ordered Group of Equivalent Games	
1. The Negative of a Game. 2. The Sum of Two Games.	
3. Isomorphic Games. 4. A Partial Ordering of Games.	
5. Equality of Games	
§6. Games and CONWAY Games	343
1. The Fundamental Mappings. 2. Extending to CONWAY Games the Definitions of the Relations and Operations Defined for Games. 3. Examples	
§7. CONWAY Numbers	346
1. The CONWAY Postulates (C1) and (C2). 2. Elementary Properties of the Order Relation. 3. Examples	
§8. The Field of CONWAY Numbers	349
1. The Arithmetic Operations for Numbers. 2. Examples.	
3. Properties of the Field of Numbers	
References	353



*Chapter 14. Set Theory and Mathematics.*  
H.-D. Ebbinghaus

355

Introduction

§1. Sets and Mathematical Objects

355

1. Individuals and More Complex Objects. 2. Set Theoretical Definitions of More Complex Objects.

3. Urelements as Sets

§2. Axiom Systems of Set Theory

363

1. The RUSSELL Antinomy. 2. ZERMELO's and the ZERMELO-FRAENKEL Set Theory. 3. Some Consequences. 4. Set Theory with Classes

§3. Some Metamathematical Aspects

372

1. The VON NEUMANN Hierarchy. 2. The Axiom of Choice. 3. Independence Proofs

Epilogue

378

References

378

Name Index

381

Subject Index

387

Portraits of Famous Mathematicians

393

# Introduction

*K. Lamotte*

Mathematics, according to traditional opinion, deals with numbers and figures. In this book we do not begin, as EUCLID began, with figures but with numbers.

Mathematical research over the last hundred years has created abstract theories, such as set theory, general algebra, and topology, whose ideas have now penetrated into the teaching of mathematics at the elementary level. This development has not been ignored by the authors of this book; indeed, they have willingly taken advantage of it in that the authors assume the reader to be familiar with the basic concepts of (naive) set theory and algebra. On the other hand, a first volume on numbers should emphasize the fact that modern research in mathematics and its applications is, to a considerable extent, linked to what was created in the past. In particular, the traditional number system is the most important foundation of all mathematics.

The book that we now present is divided into three parts, of which the first, which may be regarded as the heart, describes the structure of the number-system, from the natural numbers to the complex and  $p$ -adic numbers. The second part deals with its further development to 'hypercomplex numbers,' while in the third part two relatively new extensions of the real number system are presented. The six chapters of the first part cover those parts of the subject of 'numbers' that every mathematician ought to have heard or read about at some time. The other two parts are intended to satisfy the appetite of a reader who is curious to learn something beyond the basic facts. On the whole, "the structure of number systems" would be a more accurate description of the content of this book.

We should now like to say a few words in more detail about the various contributions, the aims that the authors have set out to achieve, and the reasons that have induced us to bring them together in the form in which they are presented here.