Quaternions: from Algebra to Computer Graphics

Matt Klassen

DigiPen Institute of Technology

Redmond, Washington

(Slides from a talk at University of Washington ~2005.)

Mathematics Curriculum at DigiPen:

Computer Science Major:
(Game-Programming Emphasis)

Baby Linear Algebra
Calculus I
Calculus II
Discrete Math
Linear Algebra
Splines and Geometric Modeling
Elective (Quaternions)

Other Electives: Numerical Analysis, Differential Geometry, Number Theory and Cryptography, Surfaces for Geometric Modeling, Differential Equations, Graph Theory, Computational Geometry

(Mathematics Major - under development)

Course Outline

- Orthogonal transforms on $\mathbb{R}^2$ and $\mathbb{R}^3$
- Basic group theory (especially the finite quaternion group, orthogonal groups, and rotation groups)
- Fields (focusing on $\mathbb{R}$, $\mathbb{C}$, and finite fields)
- Real Algebras (Basis construction, Normed division algebras)
- S^1 (unit complex numbers) and interpolation of 2D rotations
- Hamilton's Quaternion Algebra $\mathbb{H}$
- S^3 (unit quaternions) and interpolation of 3D rotations
- Smooth Interpolation Techniques

Hamilton's Quaternion Algebra

$\mathbb{H} =$ real vector space with basis $\{1, i, j, k\}$.

Hamilton's multiplication rules

$$i^2 = j^2 = k^2 = -1 = ijk$$

turn $\mathbb{H}$ into a real associative (noncommutative) algebra with identity.

Basis product table:

$*$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Important Subspaces:

$Re(\mathbb{H}) = Span(\{1\})$

$Im(\mathbb{H}) = Span(\{i, j, k\})$

Quaternion product:

Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k = a_1 + \mathbf{v}_1$,

and $q_2 = a_2 + b_2 i + c_2 j + d_2 k = a_2 + \mathbf{v}_2$,

with $\mathbf{v}_1, \mathbf{v}_2 \in Im(\mathbb{H})$.

Then in terms of familiar dot and cross products:

$q_1 q_2 = (a_1 + \mathbf{v}_1)(a_2 + \mathbf{v}_2)$

$= (a_1 a_2 - \mathbf{v}_1 \bullet \mathbf{v}_2) + (a_1 \mathbf{v}_2 + a_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$

$=$ (Real Part) $+$ (Imaginary Part).

Special Products

From the formula:

$pq = (a + \mathbf{u})(b + \mathbf{v})$

$$= ab - \mathbf{u} \bullet \mathbf{v} + a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v}$$

we get for imaginary quaternions:

$\mathbf{uv} = -\mathbf{u} \bullet \mathbf{v} + \mathbf{u} \times \mathbf{v}$.

Also, for $p, q \in \mathbb{H}$:

$pq - qp = 2\mathbf{u} \times \mathbf{v} = \mathbf{uv} - \mathbf{vu}$.

Corollary: The center of $\mathbb{H}$ is $\mathbb{R}$, ie.

$$Z(\mathbb{H}) = \{q \in \mathbb{H} : qp = pq \ \forall p \in \mathbb{H}\} = Re(\mathbb{H}).$$

Conjugate and Norm

For $q = a + \mathbf{v} \in \mathbb{H}$, let $\overline{q} = a - \mathbf{v}$,

and let $|q|$ be the standard Euclidean norm.

Inverses: For $q \neq 0$, $q^{-1} = \frac{\overline{q}}{|q|^2}$

Thus $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$, is a multiplicative group

For any $p,q \in \mathbb{H}$: $\overline{pq} = \overline{q}\,\overline{p}$.

Corollary 1: $|pq| = |p||q|$ (the norm is multiplicative).

Corollary 2: $\mathbb{H}$ has no zero divisors.

Corollary 3: $S^3 = \{q \in \mathbb{H} : |q| = 1\}$ is a subgroup of $\mathbb{H}^*$.

Corollary 4: $\mathbb{H}$ is a *normed division algebra*. (a finite-dimensional normed real algebra, without zero divisors, and with multiplicative norm.)

Historical Aside

Hamilton was in fact looking for a 3-dimensional normed division algebra (off and on for about ten years!) He was hoping to generalize the complex numbers, with their applications to 2D problems, to 3D.

He realized that he needed the fourth dimension while walking along the Royal Canal in Dublin, on October 16th, 1843. Of that moment he writes:

"I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, and k, exactly such as I have used them ever since."

He then proceeded, in a famous act of mathematical vandalism, to carve these equations into the stone of the Brougham Bridge:

$$i^2 = j^2 = k^2 = ijk = -1.$$

More Historical Notes:

Hamilton spent the rest of his life obsessed with the quaternions and their applications to geometry.

They were fashionable for a time: They were a mandatory examination topic at Dublin University, and in some American Universities were the only advanced mathematical topic taught.

A school of 'quaternionists' developed, emphasizing applications to physics.

Meanwhile, vector geometry flourished, and a war of polemics ensued, with such luminaries as Kelvin and Heaviside weighing in on the side of vectors.

Ultimately, the quaternionists lost, and the subject acquired a slight tint of disgrace from which it has never quite recovered.

Modern Views

There are exactly four normed division algebras: (over the real numbers)

the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.

The 'Dickson Doubling Process' constructs each of these algebras by doubling the dimension.

For example: complex numbers are represented as pairs of reals: $a + bi$, and quaternions can be represented as pairs of complex numbers: $(a + bi) + (c + di)j$, etc.

Although they have not become a standard college mathematics topic, the quaternions (and also octonions) are being used increasingly in physics.

(see John Baez's web site)

Orthogonal Groups

Let V be a real inner product space of dimension n. The orthogonal group $O(V)$, the special orthogonal group $SO(V) = O^+(V)$, and its coset $O^-(V)$ are defined as follows:

$$O(V) = \{f : V \longrightarrow V \ : \ |f(\mathbf{v})| = |\mathbf{v}|, \text{ for all } \mathbf{v} \in V\}$$

$$O^+(V) = \{f \in O(V) \ : \ det(f) = 1\} = SO(V),$$

and

$$O^-(V) = \{f \in O(V) \ : \ det(f) = -1\}.$$

Generation Theorem for $O(V)$:

Let V be a real inner product space of dimension n, and let $f \in O(V)$. Then f can be written as a product of k reflections, with $k \leq n$, and k even if $f \in SO(V)$.

Cayley's Theorem:

Let $\phi \in O(\mathbb{H})$. Then there exist unit quaternions $q_1, q_2 \in S^3$ such that

$$\phi(x) = q_1 x q_2, \quad \text{and} \quad \phi \in O^+(\mathbb{H})$$

or

$$\phi(x) = q_1 \overline{x} q_2, \quad \text{and} \quad \phi \in O^-(\mathbb{H}).$$

Hamilton's Theorem:

Let $\phi \in O(Im(\mathbb{H}))$. Then there exists a $q \in S^3$ such that

$$\phi(x) = q x \overline{q} \quad \text{and} \quad \phi \in O^+(Im(\mathbb{H}))$$

or

$$\phi(x) = -q x \overline{q} \quad \text{and} \quad \phi \in O^-(Im(\mathbb{H})).$$

Quaternion Rotation Operator

First, let $R_{\mathbf{u},\theta}$ be the rotation on $\mathbb{R}^3$

about the axis (unit) vector $\mathbf{u}$ through an angle θ

(counterclockwise according to the right hand rule.)

Now for $q \in S^3$, $q \neq \pm 1$, we can write uniquely:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{u}$$

where $\mathbf{u}$ is a unit vector in $Im(\mathbb{H})$, and $\theta \in (0, 2\pi)$.

Then the function $R_q(x) = qx\overline{q}$, for $x \in Im(\mathbb{H})$

coincides with the function $R_{\mathbf{u},\theta}$.

Note: $R_q(x) = R_{-q}(x)$.

This induces a 2-1 map:

$$\begin{aligned} S^3 &\longrightarrow SO(3) \\ \{q, -q\} &\mapsto R_{\mathbf{u},\theta} \end{aligned}$$

Interpolation of Rotations

Problem: Given rotation matrices $A_1, \ldots, A_k$ in $SO(3)$, find a smooth path (curve) in $SO(3)$ which interpolates the A_i.

It is not straightforward to work with parametrized curves directly with the matrix representation, or directly in $SO(3)$.

An elegant solution is to pull back to S^3 under the 2-1 map $S^3 \longrightarrow SO(3)$.

An interpolation path is found on S^3, and then the image on $SO(3)$ is used to interpolate the rotation matrices.

Applications to Computer Graphics and Animation

In traditional animation, key-frames are specified by an animator, and then in-between frames are supplied by other animators.

In computer graphics in-between frames can be generated automatically using a variety of blending and interpolation techniqes.

Similarly, an object in 3D may have certain key-orientations in a sequence. The computer animator may then want to insert in-between orientations.

Each orientation corresponds to a 3×3 rotation matrix, which in turn can be pulled back to a unit quaternion.

Thus smooth interpolation of unit quaternions yields smooth interpolation of orientations in 3D.

Specific Interpolation Paths

1) C^0 Spherical Linear Interpolation (Slerp)

Between each pair of quaternions a great circular arc on S^3 is used.

2) C^1 Cubic Bezier Curve Analog

Nested Spherical Linear Interpolation replaces the nested linear interpolation of De Casteljau's algorithm. Cubic-type curves are splined together.

3) C^2 Circular Arc Blending

Through each triple of quaternions a circular arc on S^3 is described. Pairs of circular arcs are then blended using Slerp, producing a C^2 spline path.

Orientations in the demo:

1. identity
2. $\mathbf{u} = (0,0,1)$, $\theta = -\pi/2$
3. $\mathbf{u} = (0,1,0)$, $\theta = \pi$
4. $\mathbf{u} = (1,0,0)$, $\theta = \pi/2$

Spherical Linear Interpolation

Let $q = \cos\theta + \sin\theta\mathbf{u} \in S^3$.

Then $Slerp(1, q, t) = \cos(\theta t) + \sin(\theta t)\mathbf{u} = q^t$.

Let $p = \cos\theta_1 + \sin\theta_1\mathbf{u}_1$, $q = \cos\theta_2 + \sin\theta_2\mathbf{u}_2 \in S^3$, and $\theta =$ the angle between p and q:

Then $Slerp(p, q, t) = \cos(\theta t)p + \sin(\theta t)r$ where p and r are a Gram-Schmidt basis:

$$r = \frac{q - proj_p q}{|q - proj_p q|}$$

Exponential Form: $Slerp(p, q, t) = p(p^{-1}q)^t$

Another commonly used form:

$$Slerp(p, q, t) = \frac{\sin(1-t)\theta}{\sin\theta}p + \frac{\sin t\theta}{\sin\theta}q.$$

Cubic Bezier Curve Analog

This is Ken Shoemake's lifting to S^3 of the de Casteljau algorithm for cubic parametric curves in Euclidean space.

For control points P_0,P_1,P_2 and P_3 we define:

$\gamma_i(t) = L(P_i, P_{i+1}, t) = (1-t)P_i + tP_{i+1}$, $i = 0, 1, 2$,

$\beta_i(t) = L(\gamma_i(t), \gamma_{i+1}(t), t)$, $i = 0, 1$,

and $\alpha(t) = L(\beta_0(t), \beta_1(t), t)$.

Then $\alpha(t)$ also has the Bernstein-Bezier form:

$$(1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3$$

For control quaternions q_0,q_1,q_2 and q_3 we define $\alpha(t)$, $\beta_i(t)$, and $\gamma_i(t)$ just as above, but with L replaced by $Slerp$.

C^1 cubic spline analog

Let P_0, P_1, and P_2 be key points. Then it is possible to insert control points Q_0, R_0 and Q_1, R_1 so that the two cubic Bezier curves given by the control points: P_0,Q_0,R_0,P_1 and P_1,Q_1,R_1,P_2 are joined smoothly.

Since the Bezier curve is tangent to its control polyline at the ends, we simply need to ensure that:

R_0,P_1 and Q_1 are collinear.

All of this lifts to S^3. We just need to make a choice of tangent direction at P_1.

Shomake's choice of control quaternions:

$Q_1 = Bisect(Double(P_0, P_1), P_2)$

$R_0 = Double(Q_1, P_1)$

where $Double(p, q) = 2(p \bullet q)q - p$ and

$$Bisect(p, q) = \frac{p + q}{|p + q|}.$$

C^2 cubic spline analog ?

Note: In Euclidean space we can do much better: the complete C^2 cubic spline which passes through a sequence of key points is found by solving a linear system in the vector space of C^2 cubic splines.

This, however, does not lift to S^3 since we no longer have a vector space of functions to work with.

Further, the variational property of the complete cubic spline does not lift to S^3, and there is no known closed formula for such a curve on S^3.

Circular Arc Blending

Each triple of distinct unit quaternions q_{i-1}, q_i, q_{i+1} determines a plane in $\mathbb{H}$, whose intersection with S^3 is a circle.

Thus each pair of quaternions q_i, q_{i+1} determines two circular paths, say $\alpha_i(t)$ and $\beta_i(t)$, reparametrized so that $\alpha_i(0) = \beta_i(0) = q_i$ and $\alpha_i(1) = \beta_i(1) = q_{i+1}$.

Now let $\gamma_i(t) = Slerp(\alpha_i(t), \beta_i(t), f(t))$, where $f(t) = t$ or $f(t) = 3t^2 - 2t^3$.

Finally, let $\gamma(t)$ be the path obtained by glueing together the pieces $\gamma_i(t)$.

Then $\gamma(t)$ is C^1 for $f(t) = t$ and C^2 for $f(t) = 3t^2 - 2t^3$.

Triple Product Identity:

$$qpq = 2(\overline{p} \bullet q)q - (q \bullet q)\overline{p}$$

Proof:

Apply $q\overline{q} = |q|^2 = q \bullet q$ to $p + q$:

We get: $(p+q)(\overline{p+q}) = (p+q) \bullet (p+q)$

$= p \bullet p + 2p \bullet q + q \bullet q.$

On the other hand, we have: $(p+q)(\overline{p+q})$

$= (p+q)(\overline{p} + \overline{q}) = p\overline{p} + p\overline{q} + q\overline{p} + q\overline{q}.$

Thus: $p\overline{q} + q\overline{p} = 2p \bullet q$

Now right multiply by q: $p\overline{q}q + q\overline{p}q = 2(p \bullet q)q$

and replace p with $\overline{p}$ to get:

$qpq = 2(\overline{p} \bullet q)q - (q \bullet q)\overline{p}$

Reflection maps

For $q \in S^3$ write:

$$r_q(x) = x - 2(q \bullet x)q$$

(reflection in the hyperplane orthogonal to q.)

example: $r_1(x) = x - 2(1 \bullet x)1 = x - 2Re(x) = -\overline{x}$

Now let $F_q(x) = qxq$ be the triple product function.

From the triple product identity we have:

$F_q(x) = 2(\overline{x} \bullet q)q - \overline{x} = -r_q(\overline{x})$

and $r_q(x) = -F_q(\overline{x}) = F_q(r_1(x)))$.

Also: $q_1, q_2 \in S^3$: $r_{q_1} \circ r_{q_2} = F_{q_1} \circ F_{\overline{q}_2}$.

Generation Theorem for $O(\mathbb{H})$:

Theorem: $O(\mathbb{H})$ is generated by the functions

$$F_q(x) = qxq,\ q \in S^3, \quad \text{and} \quad r_1(x) = -\overline{x}.$$

Further, if $\phi \in SO(\mathbb{H})$, then ϕ is a product of 2 or 4 maps F_q, $q \in S^3$.

Proof: Use $r_q = F_q \circ r_1$, together with the generation theorem for $O(V)$ to see that $O(\mathbb{H})$ is generated by the F_q, $q \in S^3$, and r_1. For the second part use $r_{q_1} \circ r_{q_2} = F_{q_1} \circ F_{\overline{q}_2}$.

Lemma: If $\psi \in O(Im(\mathbb{H}))$, then ψ extends uniquely to $\widehat{\psi} \in O(\mathbb{H})$ such that $\psi \in O^+(Im(\mathbb{H})) \iff \widehat{\psi} \in O^+(\mathbb{H})$.

Proof: Define $\widehat{\psi}(x) = x$ if $x \in Re(\mathbb{H})$, and $\widehat{\psi}(x) = \psi(x)$ if $x \in Im(\mathbb{H})$. Then if $q = a + \mathbf{v}$ we define $\widehat{\psi}(a + \mathbf{v}) = a + \psi(\mathbf{v})$. One can see from the form of the matrix of $\widehat{\psi}$ that $det(\widehat{\psi}) = det(\psi)$.

Proof of Cayley's Theorem

Cayley's Theorem:

Let $\phi \in O(\mathbb{H})$. Then there exist unit quaternions $q_1, q_2 \in S^3$ such that

$$\phi(x) = q_1 x q_2, \quad \text{and} \quad \phi \in O^+(\mathbb{H})$$

or

$$\phi(x) = q_1 \overline{x} q_2, \quad \text{and} \quad \phi \in O^-(\mathbb{H}).$$

Proof: If $\phi \in O^+$ then $\phi(x) = (p_2p_1)x(p_1p_2)$ or $\phi(x) = (p_4p_3p_2p_1)x(p_1p_2p_3p_4)$. In either case, $\phi(x) = q_1 x q_2$, with some $q_1, q_2 \in S^3$. If $\phi \in O^-$ then $\phi \circ r_1 \in O^+$, hence $\phi(-\overline{x}) = \phi \circ r_1(x) = p_1 x p_2$, for some $p_1, p_2 \in S^3$, by part i). Then $\phi(x) = p_1(-\overline{x})p_2 = (-p_1)\overline{x}p_2$.

Proof of Hamilton's Theorem:

Hamilton's Theorem:

Let $\phi \in O(Im(\mathbb{H}))$. Then there exists a $q \in S^3$ such that

$$\phi(x) = qx\overline{q} \quad \text{and} \quad \phi \in O^{+}(Im(\mathbb{H}))$$

or

$$\phi(x) = -qx\overline{q} \quad \text{and} \quad \phi \in O^{-}(Im(\mathbb{H})).$$

Proof: Let $\phi \in O^{+}(Im(\mathbb{H})))$. Then $\widehat{\phi} \in O^{+}(\mathbb{H})$, so $\widehat{\phi}(x) = q_1xq_2$ with $q_1,\ q_2 \in S^3$. From $\widehat{\phi}(a + \mathbf{v}) = a + \phi(\mathbf{v})$, we have $\widehat{\phi}(1) = 1$. So $q_1 1 q_2 = 1 \rightarrow q_2 = q_1^{-1}$. So $\widehat{\phi}(x) = qx\overline{q} \Rightarrow \phi(x) = qx\overline{q}$.

If $\phi \in O^{-}(Im(\mathbb{H})))$, then by checking determinants we have $-\phi \in O^{+}(Im(\mathbb{H})))$. So $\phi(x) = -qx\overline{q}$.

Axis and Angle

For $q = a + \mathbf{v} \in S^3$:

$R_q = I$ (identity) $\Leftrightarrow q = \pm 1$.

If $R_q \neq I$ then $0 \neq q - \overline{q} = 2\mathbf{v} \in Im(\mathbb{H})$, and

$R_q(2\mathbf{v}) = R_q(q - \overline{q}) = qq\overline{q} - q\overline{q}\,\overline{q} = q - \overline{q}$.

Thus the line $\mathbb{R}\mathbf{v}$ is fixed by R_q.

For $q \in S^3 \setminus \{\pm 1\}$, $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{u}$

with $|\mathbf{u}| = 1$ and $\theta \in (0, 2\pi)$ is unique.

Further, for $\mathbf{x} \in Im(\mathbb{H})$ we can write

$$R_q(\mathbf{x}) = \cos\theta\mathbf{x} + \sin\theta\mathbf{u} \times \mathbf{x} + (1 - \cos\theta)(\mathbf{u} \bullet \mathbf{x})\mathbf{u}$$

for all $\mathbf{x} \in Im(\mathbb{H})$.

Proof: Let $\alpha = \cos\frac{\theta}{2}$ and $\beta = \sin\frac{\theta}{2}$.

Then: $\alpha^2 - \beta^2 = \cos\theta$, $2\alpha\beta = \sin\theta$,
and $2\beta = 1 - \cos\theta$.

So $R_q(\mathbf{x}) = (\alpha + \beta\mathbf{u})\mathbf{x}(\alpha - \beta\mathbf{u})$

$= \alpha^2\mathbf{x} + \alpha\beta(\mathbf{ux} - \mathbf{xu}) - \beta^2\mathbf{uxu}$

$= \alpha^2\mathbf{x} + \alpha\beta(2\mathbf{u} \times \mathbf{x}) - \beta^2(2(\overline{\mathbf{x}} \bullet \mathbf{u})\mathbf{u} - (\mathbf{u} \bullet \mathbf{u})\overline{\mathbf{x}})$,

(from the triple product identity)

$= \alpha^2\mathbf{x} + 2\alpha\beta(\mathbf{u} \times \mathbf{x}) - \beta^2(\mathbf{x} - 2(\mathbf{x} \bullet \mathbf{u})\mathbf{u})$

$= (\alpha^2 - \beta^2)\mathbf{x} + 2\alpha\beta(\mathbf{u} \times \mathbf{x}) + 2\beta^2(\mathbf{x} \bullet \mathbf{u})\mathbf{u}$

$= \cos\theta\mathbf{x} + \sin\theta\mathbf{u} \times \mathbf{x} + (1 - \cos\theta)(\mathbf{u} \bullet \mathbf{x})\mathbf{u}$.

Corollary: If $|\mathbf{x}| = 1$ and $\mathbf{u} \bullet \mathbf{x} = 0$, then:

$$R_q(\mathbf{x}) \bullet \mathbf{x} = \cos\theta.$$

Reference:

M. Koecher and R. Remmert:
"Hamilton's Quaternions",
(Chapter 7 in the book "Numbers")
Springer GTM 123.