MAT 364/564 Example for HW 1 Fall 2020

In this example we illustrate the necessary details of a proof.

Problem: Show that the game of two n by n domineering boards can always be won by the second player. Assume that n is a positive integer, and the game consists of playing on the two boards simultaneously, and that the boards are not connected. Each player takes a turn by making a legal domineering move on either board. The usual end condition applies: the player that cannot make a legal move loses the game.

Proof: We need to show that for all n the second player has a winning strategy. In the case $n = 1$ neither player can make a move, so the first player immediately loses. Thus the second player wins. (The winning strategy here is to simply play the game.) For $n \geq 2$ it is possible for first player to make a move. After first player makes a move, the second player should follow a symmetry strategy which always guarantees that if after this the move it is still possible for first player to move again, then it will always be possible for the second player move again. The way to accomplish this is to also guarantee that the two boards are always equivalent by a rotation of 90 degrees.

Label the boards B1 and B2, and the players First and Second. If First moves on B1, then second should move on B2 by rotating B1 by 90 degrees clockwise and transferring the move that was just made by First onto B2. Note that this move will be guaranteed to be a legal move for Second, since Left and Right make moves which are related by a 90 degree rotation. This move will also exist on B2 since B2 is identical to B1 by a 90 degree rotation. If First moves on B2 then Second should move on B1 in a similar way but by rotating B2 90 degrees counterclockwise and then copying the move made by first onto B1. The rotations of boards are only made in the mind of Second, not actually physically affecting the boards or the game state.

By making the moves according to the above strategy, Second maintains that the boards are equivalent by a 90 degree rotation, and that if First can move then Second can also make a symmetric move in response. Thus Second has a winning strategy.