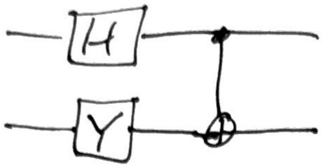
MAT 399 Midterm Exam

1. a)



$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(i\,|1\rangle)$$
$$\overset{=}{|0\rangle|0\rangle} \qquad = \frac{i}{\sqrt{2}}(|01\rangle + |11\rangle) \mapsto \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|01\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(-i\,|0\rangle)$$
$$\overset{=}{|0\rangle|1\rangle} \qquad = -\frac{i}{\sqrt{2}}(|00\rangle + |10\rangle) \mapsto -\frac{i}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(i\,|1\rangle)$$
$$\overset{=}{|1\rangle|0\rangle} \qquad = \frac{i}{\sqrt{2}}(|01\rangle - |11\rangle) \mapsto \frac{i}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$|11\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(-i\,|0\rangle)$$
$$\overset{=}{|1\rangle|1\rangle} \qquad = -\frac{i}{\sqrt{2}}(|00\rangle - |10\rangle) \mapsto -\frac{i}{\sqrt{2}}(|00\rangle - |11\rangle)$$

before CNOT: $\dfrac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

after CNOT: $A = \dfrac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

1.b) $$\bar{A}^T = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad A = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\bar{A}^T A = -\frac{1}{2}\begin{pmatrix} -2 & & & 0 \\ & -2 & & \\ & & -2 & \\ 0 & & & -2 \end{pmatrix} = I \quad \checkmark$$

2. prob. of meas. $|00\rangle$ if input $|00\rangle$ is 0.
3. — ——— $|10\rangle$ $= 1/2$

4.
$$\begin{aligned}
|000\rangle &\mapsto \tfrac{i}{\sqrt{2}}(|01\rangle + |10\rangle)|0\rangle \xmapsto{\text{(CCNOT)}} \tfrac{i}{\sqrt{2}}(|010\rangle + |100\rangle) \\
|001\rangle &\mapsto \quad '' \quad |1\rangle \mapsto \tfrac{i}{\sqrt{2}}(|011\rangle + |101\rangle) \\
|010\rangle &\mapsto -\tfrac{i}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle \mapsto -\tfrac{i}{\sqrt{2}}(|000\rangle + |111\rangle) \\
|011\rangle &\mapsto \quad '' \quad |1\rangle \mapsto -\tfrac{i}{\sqrt{2}}(|001\rangle + |110\rangle) \\
|100\rangle &\mapsto \tfrac{i}{\sqrt{2}}(|01\rangle - |10\rangle)|0\rangle \mapsto \tfrac{i}{\sqrt{2}}(|010\rangle - |100\rangle) \\
|101\rangle &\mapsto \quad '' \quad |1\rangle \mapsto \tfrac{i}{\sqrt{2}}(|011\rangle - |101\rangle) \\
|110\rangle &\mapsto -\tfrac{i}{\sqrt{2}}(|00\rangle - |11\rangle)|0\rangle \mapsto -\tfrac{i}{\sqrt{2}}(|000\rangle - |111\rangle) \\
|111\rangle &\mapsto \quad '' \quad |1\rangle \mapsto -\tfrac{i}{\sqrt{2}}(|001\rangle - |110\rangle)
\end{aligned}$$

4. a)

$$A = \frac{i}{\sqrt{2}} \cdot \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

b)

$$\bar{A}^T = -\frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

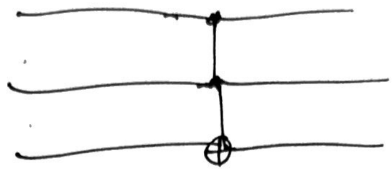
Check $A\bar{A}^T = I$. ✓

5. prob. = 0

6. "

7. over →

7. CCNOT:



$$|1000\rangle \mapsto |1000\rangle$$
$$|1001\rangle \mapsto |1001\rangle$$
$$|1010\rangle \mapsto |1010\rangle$$
$$|1011\rangle \mapsto |1011\rangle$$
$$|1100\rangle \mapsto |1100\rangle$$
$$|1101\rangle \mapsto |1101\rangle$$
$$|1110\rangle \mapsto |1111\rangle$$
$$|1111\rangle \mapsto |1110\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) f :

000	$\mapsto$ 0
001	1
010	0
011	1
100	0
101	1
110	1
111	0

f is balanced.

b) by appending gates to the right of CCNOT the resulting circuit will have a unitary matrix U. The resulting function g will then be computed by the last digit of the output of U. But U is 1-1, so g will be balanced.

8. a) $\{00, 01, 10, 11\} = \{0,1\}^2 = \mathbb{F}_2^2$

shift string is $s = 01$

$00 \mapsto$ any of the 4, say x : (4 choices)
$01 \mapsto$ also goes to x
$10 \mapsto$ any y of the remaining 3 (3 choices)
$11 \mapsto y$

So there should be 12 such f
List as one column, but only of height 2:

00	00	00	01	01	01	10	10	10	11	11	11
01	10	11	00	10	11	00	01	11	00	01	10

b) orthogonal to 01 :

$s^{\perp} = \{00, 10\}$

c) $f(00) = 10$
$\quad\; 01 \quad 10$
$\quad\; 10 \quad 00$
$\quad\; 11 \quad 00$

$\mathcal{U}_f : \mathbb{H}_2 \otimes \mathbb{H}_2 \longrightarrow \mathbb{H}_2 \otimes \mathbb{H}_2$

$|x\rangle|y\rangle \mapsto |x\rangle|f(x)\oplus y\rangle$

$|00\rangle|00\rangle \mapsto |00\rangle|10\rangle$
$|00\rangle|01\rangle \mapsto |00\rangle|11\rangle$
$|00\rangle|10\rangle \mapsto |00\rangle|00\rangle$
$|00\rangle|11\rangle \mapsto |00\rangle|01\rangle$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = B$$

$|01\rangle|00\rangle \mapsto |01\rangle|10\rangle$
$|01\rangle|01\rangle \mapsto |01\rangle|11\rangle$
$|01\rangle|10\rangle \mapsto |01\rangle|00\rangle$
$|01\rangle|11\rangle \mapsto |01\rangle|01\rangle$

$\left(\quad '' \quad\right)$

$|10\rangle|00\rangle \mapsto |10\rangle|00\rangle$
$|10\rangle|01\rangle \mapsto |10\rangle|01\rangle$
$|10\rangle|10\rangle \mapsto |10\rangle|10\rangle$
$|10\rangle|11\rangle \mapsto |10\rangle|11\rangle$

I

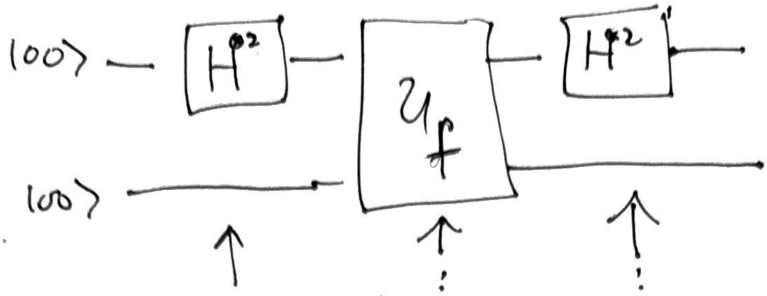
etc.

$\left('' \right)$

Block form of matrix 16×16, blocks of size 4×4.

$$\begin{pmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

8. d)



$$\tfrac{1}{2}(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)|00\rangle = \tfrac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)|00\rangle$$

$$\tfrac{1}{2}\left(|00\rangle|10\rangle + |01\rangle|10\rangle + |10\rangle|00\rangle + |11\rangle|00\rangle\right)$$

$$\begin{aligned}
= \tfrac{1}{2}\Big[& \tfrac{1}{2}(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)|10\rangle \\
& + \tfrac{1}{2}(|0\rangle+|1\rangle)(|0\rangle-|1\rangle)|10\rangle \\
& + \tfrac{1}{2}(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)|00\rangle \\
& + \tfrac{1}{2}(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)|00\rangle \Big]
\end{aligned}$$

$$\begin{aligned}
= \tfrac{1}{4}\Big[& \left(\underline{|00\rangle} + \cancel{|01\rangle} + \underline{|10\rangle} + \cancel{|11\rangle} + |00\rangle - \cancel{|01\rangle} + |10\rangle - \cancel{|11\rangle}\right)|10\rangle \\
& + \left(\underline{|00\rangle} + \cancel{|01\rangle} - \underline{|10\rangle} - \cancel{|11\rangle} + |00\rangle - \cancel{|01\rangle} - |10\rangle + \cancel{|11\rangle}\right)|00\rangle
\end{aligned}$$

$$= \tfrac{1}{2}\Big[\left(|00\rangle + |10\rangle\right)|10\rangle + \left(|00\rangle - |10\rangle\right)|00\rangle \Big]$$

This is output just before measurement.

e) prob of $|00\rangle|10\rangle$, $|10\rangle|10\rangle$, $|00\rangle|00\rangle$, $|10\rangle|00\rangle$ are all 1/4, others are all 0.