MAT 399 Quantum Algorithms Fall 2025

Final Exam

Due date: Wednesday, December 10, midnight, on Moodle

Work the problems on paper on your own. Use of AI is not permitted for this exam. Scan your work as pdf and submit on Moodle. You may use Qiskit for any parts of this exam. If you use it, include a copy of your code and output in your submission.

Let $\mathbb{H}_2$ be the two qubit space with standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and define: $|s_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$, $|s_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$ and $|s\rangle = \frac{1}{2}(\{|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Let P be the plane in $\mathbb{H}_2$ with orthogonal basis $S = \{\{|s_0\rangle, |s_1\rangle\}$. Let U_1 be the reflection operator across the hyperplane orthogonal to $|s_1\rangle$. Let U_s be the reflection operator across $|s\rangle$. Let f be function which takes bit strings of length two to the set $\{0,1\}$ with the property that for any such bit string x we have $f(x) = 0$ only if $|x\rangle$ has nonzero coefficient in $|s_0\rangle$, and $f(x) = 1$ only if $|x\rangle$ has nonzero coefficient in $|s_1\rangle$. Let $\alpha = \pi/12$ and $|\psi\rangle = \cos\alpha\,|s_0\rangle + \sin\alpha\,|s_1\rangle$.

1. Write U_1 in terms of an outer product of kets and the identity matrix.
2. Find the 4×4 matrix of U_1 on $\mathbb{H}_2$.
3. Find $U_1|s\rangle$ in the basis for $\mathbb{H}_2$.
4. Find the 2×2 matrix of U_1 restricted to P with respect to S.
5. Write U_s in terms of an outer product of kets and the identity matrix.
6. Find the 4×4 matrix of U_s on $\mathbb{H}_2$.
7. Find $U_s|s_1\rangle$
8. Find the 2×2 matrix of U_s restricted to P with respect to S.
9. Let $G = U_sU_1$. Find the matrix for G as an operator on $\mathbb{H}_2$.
10. Find $G|s\rangle$ in the basis for $\mathbb{H}_2$.
11. Find $G|s\rangle$ in the basis S for P.
12. Find θ so that $|s\rangle = \cos\theta\,|s_0\rangle + \sin\theta\,|s_1\rangle$.
13. Find $U_1|\psi\rangle$ and $G|\psi\rangle$ (the first Grover iterate applied to $|\psi\rangle$).
14. How many iterations of G, applied to $|\psi\rangle$, does it take to get as close as possible to $|s_1\rangle$? (Give a positive integer answer and say why.)
15. Use the quantum circuit in Figure 7.1.6 to compute $-U_s$ on each of the basis states of $\mathbb{H}_2$. For each one, give the inputs and the intermediate state after the controlled Z and then the final state.