MAT 399 Quantum Algorithms Fall 2025

Midterm Exam

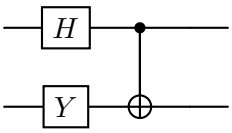
Due date: Friday, October 24, midnight, on Moodle

Work the problems on paper on your own. Use of AI is not permitted for this exam. Scan your work as pdf and submit on Moodle. You may use Qiskit for any parts of this exam. If you use it, include a copy of your code and output in your submission.

1. Consider the quantum circuit below which applies a Hadamard gate to the first qubit followed by a Pauli Y gate applied to the second qubit, then a standard CNOT gate with first qubit control and second qubit target such that the second qubit is flipped only when the first qubit equals 1.

   (a) Find the 4×4 complex matrix of this circuit with respect to the computational basis.

   (b) Show that this matrix is unitary.



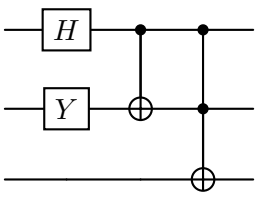
2. Same circuit as in the previous problem. If both input qubits are set to $|0\rangle$, and the final output is measured on both wires in the computational basis, what is the probability that the measurement will yield the basis state $|00\rangle$?

3. Same circuit as in the previous problem. If both input qubits are set to $|0\rangle$, and the final output is measured on both wires in the computational basis, what is the probability that the measurement will yield the basis state $|10\rangle$?

4. Use the circuit from the previous problem on the first two wires of a 3 qubit circuit then add one gate to the end of the circuit, which is a CCNOT gate using the first two qubits as controls (if both are 1 then the target is flipped) and the third qubit as the target.

   (a) Find the 8×8 complex matrix of this circuit with respect to the computational basis.

   (b) Show that this matrix is unitary.



5. Same circuit as in the previous problem. If all 3 input qubits are set to $|0\rangle$, and the final output is measured on all wires in the computational basis, what is the probability that the measurement will yield the basis state $|000\rangle$?

6. Same circuit as in the previous problem. If all 3 input qubits are set to $|0\rangle$, and the final output is measured on all wires in the computational basis, what is the probability that the measurement will yield the basis state $|111\rangle$?

7. Suppose that $f : \{0,1\}^3 \longrightarrow \{0,1\}$, is computed from a standard CCNOT circuit acting on the computational basis of $\mathbb{H}_3 = \mathbb{H}_1 \otimes \mathbb{H}_1 \otimes \mathbb{H}_1$ by using input string abc converted to input $|a\rangle\,|b\rangle\,|c\rangle$ and using the output on the third qubit wire $|y\rangle$ so that $f(abc) = y$.

   (a) Determine if this f is constant or balanced, or neither.

   (b) By appending blocks of X's and CCNOT gates (of all types), attempt to convert the standard CCNOT circuit to give new functions g_1 and g_2 of each of the other two types. If you believe it is not possible to produce one or both of these functions, give your argument for why this is true. If you succeed in producing g_1 and g_2, in the end you should have 3 functions, one of which is constant, one is balanced, and one is neither, computed from circuits which satisfy the description above. (Note: the additional gates to the right of the CCNOT can use any number of X's on any of the wires interspersed in any way with CCNOT gates of any type.)

8. Let $f : \{0,1\}^2 \longrightarrow \{0,1\}^2$ be a Simon oracle function, with shift string $s = 01$. (Recall: f is required to give the same output on inputs $|x\rangle$ and $|x \oplus s\rangle$, where x is any bit string of length 2.)

   (a) How many functions f are there which satisfy this description? (f needs to be a Simon oracle function with shift string $s = 01$.)

   (b) Write all elements of the set $s^{\perp}$ which is the subspace of the vector space $\mathbb{F}^2$ of vectors orthogonal to s.

   (c) Assume $f(00) = 10$ and $f(10) = 00$. Find the matrix of the circuit U_f.

   (d) Find the output of the first 2-qubit register in Simon's circuit just before measurement.

   (e) What are the probabilities of each basis state in the final output of Simon's circuit after measurement?