MUS 470/470L Homework 2

Fall 2018

Due date: Tuesday, November 6.

Option A:

1. Implement the 32 linear truncated icosahedral basis splines in code.
2. Graph a linear combination of these using any 3D graphing software API.
3. Design a simple UI which allows the user to change the coefficients of the 32 functions

Option B:

1. Implement the 92 continuous linear triangulated truncated icosahedral basis splines in code.
2. Graph a linear combination of these using any 3D graphing software API.
3. Design a simple UI which allows the user to change the coefficients of the 92 functions
4. Approximate the graphs of the microphone types in the first project using these functions

Linear spline functions on a truncated icosahedron

First, a *regular icosahedron* is a 3D object made up of 20 faces, 30 edges, and 12 vertices. Each face is an equilateral triangle with edge length a. All the vertices lie on a sphere of radius

$$r = a \sin \frac{2\pi}{5}.$$

(For further details, see the first Wiki reference below.)

A *truncated icosahedron*, which we will abbreviate $\mathcal{T}$, is obtained from the regular icosahedron by replacing each vertex by a regular pentagon. This is done by reducing each edge by one third of its length at each end. Each vertex is thus replaced by 5 new vertices, giving a total of 60 vertices on this new object. The faces of this object are then 12 pentagons, with the remaining 20 truncated faces (formerly regular triangles) now being regular hexagons, for a total of 32 faces. Since each pentagon introduces 5 new edges, we have the 30 truncated edges, plus another 60, giving 90 edges.

Coordinates of the vertices, with center at the origin, can be given with the constant ϕ, the Golden Mean,

$$\phi = \frac{1+\sqrt{5}}{2}$$

as follows:

$$(0, \pm 1, \pm 3\phi), (\pm 1, \pm 3\phi, 0), (\pm 3\phi, 0, \pm 1) \quad (12)$$

$$(\pm 1, \pm(2+\phi), \pm 2\phi), (\pm(2+\phi), \pm 2\phi, \pm 1), (\pm 2\phi, \pm 1, \pm(2+\phi)) \quad (24)$$

and

$$(\pm\phi, \pm 2, \pm\phi^3), (\pm 2, \pm\phi^3, \pm\phi), (\pm\phi^3, \pm\phi, \pm 2) \quad (24)$$

For any face on a truncated icosahedron, we will refer to the adjacent faces either as *surrounding faces* of F, or as *neighbors* of F. We may also refer to the edge joining two adjacent faces F and H as $F \cap H$, the intersection of F and H.

A function f on $\mathcal{T}$ is a function in three variables x, y, z which can be evaluated on each of the faces of $\mathcal{T}$. We will consider functions from $\mathcal{T}$ to $\mathbb{R}^3$.

One such simple function f_k is a scaling function which takes each point P on $\mathcal{T}$ to the constant multiple kP where P is considered as the endpoint of a vector in $\mathbb{R}^3$.

For each face F of $\mathcal{T}$, we can construct a function $f_{k,F}$ which is given by f_k on the face F, and is the identity function on all other faces. This function is clearly discontinuous on the joining edges from F to the surrounding faces.

Next, construct a function $g_{k,F}$ which is also f_k on the face F, but has a different output on the faces attached to F. In the case where F is a pentagon, the surrounding faces are all hexagons. On one hexagon, say H, label the edge E_1 of H adjacent to F, and label the edge E_2 of H parallel to E_1. Let the scaled version of E_1 by f_k be called $f(E_1)$. Now let M be the plane in $\mathbb{R}^3$ through the two edges $f(E_1)$ and E_2. We can now define the values of $g_{k,F}(x, y, z)$ for some point (x, y, z) on H to be the intersection point of the line through the origin and (x, y, z) with the plane M. $g_{k,F}$ is defined in this same way for any of the hexagons surrounding F, and is defined to be the identity function on all remaining faces. Finally, if F is a hexagon, then it is surrounded by both hexagons and pentagons. We can use the same construction of $g_{k,F}(x, y, z)$ with the slight modification that the plane M for an adjacent pentagon will now be the plane through $f(E_1)$ and the opposite vertex V on the adjacent pentagon.

Clearly the function $g_{k,F}$ is continuous on the edges of face F.

Exercise 1:

1. Show that in the case that F is a pentagon, $g_{k,F}$ is continuous on the edges emanating out from the vertices of F, and also along the edge E_2.
2. Show that in the case that F is a pentagon, $g_{k,F}$ is *discontinuous* along the two closest edges of any neighboring pentagon, and find the maximum separation value (or jump in the function values).
3. Investigate the truth of the previous two parts in the case that F is a hexagon.

Definition of the truncated icosahedral basis splines

For each face F of the truncated icosahedron, there are either five or three surrounding hexagons, depending on whether F is a pentagon or a hexagon, respectively. In either case, each such surrounding hexagon H has exactly one other neighboring face, say K, which is directly opposite F, so that the edges $H \cap K$ and $H \cap F$ are parallel. With this understanding, we can define the linear piecewise function on $\mathcal{T}$, for any face F, to be the linear spline basis function B_F given by the function $g_{k,F}$ with the choice of k being determined in such a way as to make each surrounding hexagon of F coplanar with its adjacent neighbor K described above.

Another way to visualize this function B_F is to think of it as the function which scales a given face F outward in such a way that the surrounding faces are also pulled outward but are kept linear and attached to F at one edge and remain attached to their original position on the opposite edge, until each surrounding hexagon is parallel to another neighboring hexagon.

Exercise 2:

1. Find the value k in the definition of B_F above.
2. Find a non-trivial linear combination of the B_{F_i} which is continuous, or show that no such function exists.

References:

1. *Wikipedia:* https://en.wikipedia.org/wiki/Regular_icosahedron click here
2. *Wikipedia:* https://en.wikipedia.org/wiki/Truncated_icosahedron click here

Notes on polynomial spline functions on the truncated icosahedron $\mathcal{T}$

The linear basis splines in the previous part are special cases of the construction of functions f on $\mathcal{T}$ whose output values, given input on a face F, lie on a carefully chosen *output plane*. This output plane is determined by the choice of $f(v_i)$ for three vertices v_i of the face F. The values $f(v_i)$ must be on the line through the origin and v_i, but cannot be equal to the origin. Once three are chosen, the output plane is determined, and all other values of f are computed as the intersection of the line determined by the origin and the input point with the output plane. By this construction, such an output plane in $\mathbb{R}^3$ will have the property that it intersects in exactly one point with each of the rays extending out from the origin and passing through a vertex of F.

With the above construction, we can define a linear spline on $\mathcal{T}$ to be a choice of output plane for each face of $\mathcal{T}$. Since there are 32 faces, we can see that such a function depends on $32 \cdot 3 = 96$ parameters. We can describe the set of all such functions as a product of 32 3-dimensional project spaces.

A natural question is to determine the dimension of the subspace of continuous linear splines on $\mathcal{T}$. This dimension is at least one, since the scaling functions f_k from the previous part qualify. Are these all the continuous functions in V?

Triangulated Truncated Icosahedron $\mathcal{T}_\triangle$

The truncated icosahedron $\mathcal{T}$ can be triangulated by inserting a new vertex in the center of each face F of $\mathcal{T}$, and extending new edges to each vertex of F. For practical reasons, also normalize each vertex so that all vertices lie on the unit sphere. Call this new object the Triangulated Truncated Icosahedron $\mathcal{T}_\triangle$.

Each vertex of $\mathcal{T}_\triangle$ has degree 6 or degree 5. There are 3 types of vertices:

1. degree 5, new vertex formed inside a pentagon
2. degree 6, new vertex formed inside a hexagon
3. degree 6, old vertex previously degree 3 on $\mathcal{T}$

The total number of vertices is now $92 = 60 + 32$ (60 old, and 32 new). The number of faces is $12 \cdot 5 + 20 \cdot 6 = 180$, and the number of edges is $90 + 12 \cdot 5 + 20 \cdot 6 = 270$.

Linear spline functions on the Triangulated Truncated Icosahedron $\mathcal{T}_\triangle$

Since each vertex on $\mathcal{T}_\triangle$ is surrounded by triangles, we can now easily define *continuous* linear piecewise functions, or splines, by scaling the value of any vertex. Let $f_{v,k}$ be the function on $\mathcal{T}_\triangle$ which maintains all vertices except v in place, but it scales the vertex v out from the origin by a factor k. Each of the edges on $\mathcal{T}_\triangle$ are also stretched in order to maintain continuity, so the output values on each of the surrounding faces are determined.

The functions $f_{v,k}$ can be used as a basis of continuous linear splines on $\mathcal{T}_\triangle$. It is then straightforward to model any other spherical functions, such as spherical harmonics, with these basis functions, achieving a linear approximation by simply using the values of the spherical harmonic function as the scalars k.

Exercise 3: Let $h_{k,T}$ be the function on $\mathcal{T}_\triangle$ which scales the triangular face T by the factor k, pulling along the edges surrounding T but leaving the neighboring vertices fixed. Show that the set of functions $h_{k,T}$, for any choice of $k > 1$, is not a linearly independent set of functions on $\mathcal{T}_\triangle$.