Lecture 10

Main Points:

- Proof of Existence and Uniqeness of Osculating polynomials
- Divided differences with derivatives
- Newton form for osculating polynomial

Definition of the osculating polynomial

Instead of matching only values of a data function, we might want to also match derivative values. In the following definition, we take repeated data values to mean that we are requiring consecutive matching of derivatives. It turns out to be best to require derivatives in sequence, without any gaps, which is also referred to as *Hermite* interpolation.

Given any *nondecreasing* sequence of real numbers $t_0 \leq t_1 \leq \cdots \leq t_d$ and a function $g(t)$ with values $g(t_i)$ at these numbers, suppose further that g is differentiable to order r_i at each t_i, where r_i is determined by $r_i = 0$ if $t_i < t_{i+1}$, and $r_i = k$ if $t_i = t_{i+1} = \cdots = t_{i+k}$ and $t_{i+k} < t_{i+k+1}$. Then define an *osculating polynomial* $p(t)$ with the data sequence $t_0, t_1, \cdots, t_d$ and data function $g(t)$ as a polynomial which satisfies:

$$p^{(j)}(t_i) = g^{(j)}(t_i) \;\; \textit{for} \;\; i = 0, \ldots, d, \;\; \textit{and} \;\; j = 0, \ldots, r_i.$$

Note: If we change the order of the sequence in such a way that equal values are still consecutive, the definition of the osculating polynomial is not affected. So we can allow changes in the order of the data as long as whenever $t_i = t_j$, with $i < j$, then also $t_i = t_k$ for all k satisfying $i < k < j$.

Existence and Uniqueness of the osculating polynomial

It is a fact that for any nondecreasing sequence $t_0 \leq t_1 \leq \cdots \leq t_d$ of real numbers, and function g, the osculating polynomial $p(t)$ with data function g and data values $t_0, t_1, \cdots, t_d$ exists as an element of P_d, and is unique.

Proof of Existence and Uniqueness of the osculating polynomial

Just as with the interpolating polynomial, we can prove the existence and uniqusness of the osculating polynomial using the standard basis and a linear system. In this case we need to specify both point values and derivative values according to the data sequence. We will see that the coefficient matrix for this linear system has Confluent Vandermonde determinant which is nonzero, again showing that the linear system has a unique solution.

The linear system may have some rows which come from equating values of a polynomial and its derivatives with values of the data function $g(t)$. Let's suppose that $t_0 = t_1 = t_2 = u_0$ and $t_3 = t_4 = u_1$.

$$\begin{array}{ccccccccccc} a_0 & + & a_1 u_0 & + & a_2 u_0^2 & + & a_3 u_0^3 & + & a_4 u_0^4 & = & g(u_0) \\ & & a_1 & + & 2a_2 u_0 & + & 3a_3 u_0^2 & + & 4a_4 u_0^3 & = & g'(u_0) \\ & & & & 2a_2 & + & 6a_3 u_0 & + & 12a_4 u_0^2 & = & g''(u_0) \\ a_0 & + & a_1 u_1 & + & a_2 u_1^2 & + & a_3 u_1^3 & + & a_4 u_1^4 & = & g(u_1) \\ & & a_1 & + & 2a_2 u_1 & + & 3a_3 u_1^2 & + & 4a_4 u_1^3 & = & g'(u_1) \end{array}$$

The corresponding augmented matrix of the linear system would then look like:

$$\begin{array}{cccccccc} 1 & u_1 & u_0^2 & u_0^3 & u_0^4 & | & g(u_0) \\ 0 & 1 & 2u_0 & 3u_0^2 & 4u_0^3 & | & g'(u_0) \\ 0 & 0 & 2 & 6u_0 & 12u_0^2 & | & g''(u_0) \\ 1 & u_1 & u_1^2 & u_1^3 & u_1^4 & | & g(u_1) \\ 0 & 1 & 2u_1 & 3u_1^2 & 4u_1^3 & | & g'(u_1) \end{array}$$

As we saw earlier, Confluent Vandermonde determinants were constructed in this way, and the determinants are nonzero as long as the sequence of values are distinct. In order to keep them separate, since the t_i values are allowed to be equal in order to signify derivative matching, we can assign values $u_0, u_1, \ldots, u_k$ to mean the distinct values appearing in the the list of data values $t_0, \ldots, t_d$. If each such value u_i appears with multiplicity m_i then we understand that if $m_i = 1$ then there is only one regular Vandermonde row for u_i, and if $m_i > 1$ then there are also some consecutive derivative rows with the value u_i. The determinant of the coefficient matrix is then nonzero by the Confluent Vandermonde product formula:

$$D(u_0^{m_1} u_2^{m_2} \cdots u_k^{m_k}) = \prod_{1 \leq i < j \leq n} (u_j - u_i)^{m_i m_j} \prod_{i=1}^{k} (m_i - 1)!! \neq 0,$$

where the double factorial means:

$$N!! = N!(N-1)!(N-2)! \cdots 2!1!.$$

This completes the proof of existence and uniqueness of the osculating polynomial.

Divided Differences for the osculating polynomial

For a nondecreasing sequence $t_0 \leq t_1 \leq \cdots \leq t_d$, and data function g, the divided differences are defined in the same way as they were for distinct values. In particular, we define the divided difference $[t_0, \ldots, t_d]g$ to be the coefficient of t^d in the osculating polynomial $p(t)$ with data values $t_0, \ldots, t_d$, and data function $g(t)$.

Recursive formula for the Divided Differences

For a nondecreasing sequence $t_0 \leq t_1 \leq \cdots \leq t_d$, and function g, the divided differences are defined using the same recursion as for the interpolating polynomial. For $t_0 < t_d$, we have:

$$[t_0, t_1, \ldots, t_d]g = \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0}, \quad \textit{for } t_0 < t_d,$$

and when $t_0 = t_d$ we have:

$$[t_0, t_1, \ldots, t_d]g = \frac{g^{(d)}(t_0)}{d!} \quad \textit{for } t_0 = t_d.$$

Newton form for the osculating polynomial

The Newton form for the osculating polynomial is identical to the one used for the interpolating polynomial, but with the new interpretation of the divided differences, as indicated above.

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Special cases of the osculating polynomial

When the t_i are all distinct, the osculating polynomial reduces to the interpolating polynomial. At the other extreme, when $t_0 = t_1 = \cdots = t_d$, we have the Taylor polynomial:

$$p(t) = g(t_0) + g'(t_0)(t - t_0) + \frac{g''(t_0)}{2!}(t - t_0)^2 + \cdots + \frac{g^{(d)}(t_0)}{d!}(t - t_0)^d$$

written in terms of the Taylor basis with parameter t_0.

Examples:

- Let $t_0 = 0$, $t_1 = 0$, $t_2 = 0$, $t_3 = 1$, and $t_4 = 1$. Find the osculating polynomial which agrees with the data function $g(t)$ for the above data values if $g(0) = 4$, $g'(0) = 3$ and $g''(0) = -2$, and $g(1) = 5$, and $g'(1) = -1$.

  We form the divided difference table (noting that $\dfrac{g''(0)}{2!} = -1$):

0	4				
		3			
0	4		-1		
		3		-1	
0	4		-2		1
		1		0	
1	5		-2		
		-1			
1	5				

  The Newton form is then

$$\begin{aligned} p(t) &= 4 + 3(t-0) - (t-0)^2 - (t-0)^3 + (t-0)^3(t-1) \\ &= 4 + 3t - t^2 - t^3 + t^3(t-1) \\ &= 4 + 3t - t^2 - 2t^3 + t^4 \end{aligned}$$

  with derivatives:

$$p'(t) = 3 - 2t - 6t^2 + 4t^3, \qquad \text{and} \qquad p''(t) = -2 - 12t + 12t^2,$$

  which can be seen to satisfy the original conditions.

- Note that the first three conditions $g(0) = 4$, $g'(0) = 2$ and $g''(0) = -1$, in the previous example amount to the construction of the quadratic Taylor polynomial:

$$p(t) = 4 + 3t - t^2.$$

  This illustrates the cumulative nature of the Newton form, since the requirement of more data simply adds more terms to the Newton form.

Second proof of existence of oscuating polynomial

In the second proof we again appeal to the recursive form and use induction. This time we need to verify the requirement about derivatives. The base case is the same as for interpolation: $d = 0$, with one data value t_0, so $p(t) = [t_0]g = g(t_0)$ is constant. For $d > 0$ the proof breaks into two cases:

a) $t_0 = t_d$ (and thus for all i: $t_0 = t_i = t_d$.)
b) $t_0 < t_d$.

In the first case we simply use the Taylor polynomial from Calculus. This conincides exactly with our definition for the osculating polynomial. This shows existence of the osculating polynomial in the first case. So now we assume $t_0 < t_d$.

For the induction step we assume that $p_0(t)$ and $p_1(t)$ are osculating polynomials with sequences $[t_0, \ldots, t_{d-1}]$ and $[t_1, \ldots, t_d]$ respectively. Then we form the same polynomial $p(t)$:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

According to the definition of the osculating polynomial, we now need to verify:

$$p^{(j)}(t_i) = g^{(j)}(t_i), \quad j = 0, \ldots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. In order to check this, we take some derivatives of $p(t)$, to get:

$$p^{(j)}(t) = \frac{t - t_0}{t_d - t_0} p_1^{(j)}(t) + \frac{t_d - t}{t_d - t_0} p_0^{(j)}(t) + j \cdot \frac{p_1^{(j-1)}(t) - p_0^{(j-1)}(t)}{t_d - t_0}.$$

Now to check that $p(t)$ works, assume that we have $t_i = t_{i+1} = \ldots = t_{i+r}$ for some i and r. Case b) from above now breaks into three cases:

i) $t_i = t_0$, ii) $t_i = t_d$ and iii) $t_0 < t_i < t_d$

For case i) we just plug $t_i = t_0$ into the derivative formula $p^{(j)}(t)$ and show that this equals $g^{(j)}(t_0)$, for $j = 0, \ldots, r$. The first two terms give us the correct value, since $p_0^{(j)}(t_0) = g^{(j)}(t_0)$ for $j = 0, \ldots, r$ since the sequence $t_i = t_{i+1} = \ldots = t_{i+r}$ of equal values is part of the sequence for $p_0(t)$. The only slightly tricky part is to show that the last term is zero. This follows from the fact that the sequence $t_1 = t_2 = \ldots = t_r$ has length $r-1$ and is inside the sequence for $p_1(t)$, so $p_1^{(j-1)}(t_0) = g^{(j-1)}(t_0)$ for $j = 1, \ldots, r$.

The second case is symmetric to the first, and the third case is easier since the the sequences for $p_0(t)$ and $p_1(t)$ both contain the equal values. This completes the existence part of the proof.

Before we proceed to the uniqueness proof, we establish an important fact about multiplicity of zeros of polynomials.

Multiplicity of zeros of polynomials

The usual notion of multiplicity of zero for a polynomial is given algebraically by the corresponding multiplicity of a factor. For example, the polynomial

$$p(t) = 5(t-1)^2(t-3)^5$$

has a zero at $t = 1$ of multiplicity 2, and a zero at $t = 3$ of multiplicity 5. We can also capture this information by using derivatives instead. In particular:

Definition: A polynomial $p(t)$ has a zero of multiplicity r at $t = c$ if $p(c) = 0$, $p'(c) = 0, \ldots, p^{(r-1)}(c) = 0$. In other words:

$$p^{(j)}(c) = 0, \quad j = 0, \ldots, r-1,$$

where $p^{(0)}(t) = p(t)$.

Second proof of uniquess of oscuating polynomial

For the uniqueness proof we suppose that there are two osculating polynomials $p(t)$ and $q(t)$, and we consider the difference

$$f(t) = p(t) - q(t).$$

Then $f(t)$ is in P_d and we also have:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - q^{(j)}(t_i) = g^{(j)}(t_i) - g^{(j)}(t_i) = 0, \quad j = 0, \ldots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. This means that the

Proof of the Newton form for osculating polynomial

In order to establish the Newton form, we

Lecture 6, Th Sep.22, 2011

Did Multiplicities of zeros, derivative form, proved lemma.

Defined the multiplicity of a zero for a function f at $t = c$ to be $r+1$, for $r \geq 0$, if $f(c) = f'(c) = f''(c) = \cdots = f^{(r)}(c) = 0$. (Of course, $f(t) = f^{(0)}(t)$.)

Give examples of polynomials and also at least one trig function like $f(t) = 1 - \cos t$ at $t = 0$ with multiple zeros. Show them what graphs of multiple zero points look like.

Then prove that a polynomial $f(t)$ has a zero of multiplicity $k \geq 1$ at $t = c$ if and only if $(t-c)^k$ is a factor of $f(t)$.

Continued with Newton form. Defined the general interpolation problem with derivatives (Hermite interpolation, or osculation), ie.

Given data $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_d$, and a data function $g(t)$ (with "enough derivatives"), does there exist a polynomial $p(t) \in P_d$ satisfying:

$$p^{(j)}(t_i) = g^{(j)}(t_i), \quad j = 0, \ldots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$? If $p(t)$ exists, it is also unique in P_d?

State answer as YES and YES! We did the case of strict inequality already, which is called simple interpolation. Now we want to do it in the Newton Form, in the general case. This proof uses induction. First note that there are two cases: a) $t_0 = t_d$ and b) $t_0 < t_d$. In case a) we just use the Taylor polynomial, so there is nothing more to check. So assume we are in case b).

Induction step: Assume above for one fewer data points (d instead of $d+1$), or degree $d-1$ (instead of d). Then choose two such sets: $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{d-1}$, and $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_d$. Associate the two solutions: $p_0(t)$ and $p_1(t)$, respectively, to the above interpolation problem now in these two cases.

Then we show existence of $p(t)$ with the formula:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

stopped here.

Lecture 7, T Sep.27, 2011

Quiz 2 today. Then continue with proof from last time. Now we need to take a few derivatives of $p(t)$, give formula for $p^{(j)}(t)$ and check that it works, to get existence. After reviewing the general setup, we took the derivatives, to get the formula

$$p^{(j)}(t) = \frac{t - t_0}{t_d - t_0} p_1^{(j)}(t) + \frac{t_d - t}{t_d - t_0} p_0^{(j)}(t) + j \cdot \frac{p_1^{(j-1)}(t) - p_0^{(j-1)}(t)}{t_d - t_0}.$$

Now the proof proceeds by induction. So we can assume that $p_0(t)$ and $p_1(t)$ satisfy the interpolaton conditions for their respective sequences. Now to check that $p(t)$ works, assume that we have $t_i = t_{i+1} = \ldots = t_{i+r}$ for some i and r. Case b) from above now breaks into three cases:

i) $t_i = t_0$, ii) $t_i = t_d$ and iii) $t_0 < t_i < t_d$

For case i) we just plug $t_i = t_0$ into the derivative formula $p^{(j)}(t)$ and show that this equals $g^{(j)}(t_0)$, for $j = 0, \ldots, r$. The first two terms give us the correct value, since $p_0^{(j)}(t_0) = g^{(j)}(t_0)$ for $j = 0, \ldots, r$ since the sequence $t_i = t_{i+1} = \ldots = t_{i+r}$ of equal values is part of the sequence for $p_0(t)$. The only slightly tricky part is to show that the last term is zero. This follows from the fact that the sequence $t_1 = t_2 = \ldots = t_r$ has length $r-1$ and is inside the sequence for $p_1(t)$, so $p_1^{(j-1)}(t_0) = g^{(j-1)}(t_0)$ for $j = 1, \ldots, r$.

The second case is symmetric to the first, and the third case is easier since the the sequences for $p_0(t)$ and $p_1(t)$ both contain the equal values. Based on a few questions, it was evident that at least a couple of students actually followed this proof.

Lecture 8, Th Sep.29, 2011

Then uniqueness comes from looking at zeros with multiplicities in the difference $p - q$ where p and q are both satisfying the interpolation conditions as above. The difference must then have total zero multiplicity more than d, so must be zero.

Now, make the definition of the operator: $[t_0, t_1, \ldots, t_d]g$. This is defined as *the coefficient of* t^d *in the interpolating polynomial* $p(t)$ *to the data sequence* $t_0, t_1, \ldots, t_d$ *for the data function* $g(t)$. Note: This is *not necessarily* the leading coefficient, since the coefficient of t^d can be zero.

The recursive property of this operator shows that it is a divided difference and can proved using the definition of $p(t)$ above, by simply taking the coefficient of t^d on both sides of the equation:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

Lecture 9, T Oct.4, 2011

First review all of the above, operator notation, cases, examples, etc.

Finally, construct the Newton form. This comes from looking at $p(t) - p_0(t)$. This difference can be factored completely since it has d roots counting multiplicities. We get:

$$p(t) - p_0(t) = C \cdot (t - t_0)(t - t_1) \cdots (t - t_d),$$

where the multiple factors are listed according to the equalities of the type $t_i = t_{i+1} = \cdots = t_r$. Those come from the lemma which equates the notion of multiplicity of zeros in terms of derivatives with the algebraic multiplicity.

Then talk about Project Part II, polynomial interpolation. Main point is that this can look bad for certain inputs. You can make the graph of a parametric polynomial interpolation shoot off the screen wildly by moving one of the points around. This effect will be mostly taken care of when we do Part III: Cubic Spline Interpolation.

Lecture 10, Th Oct.6, 2011

Today started with the derivation of the Leibniz Rule for divided differences in the case $d = 2$, with $f = gh$:

$$[t_0, t_1, t_2]f = [t_0]g[t_0, t_1, t_2]h + [t_0, t_1]g[t_1, t_2]h + [t_0, t_1, t_2]g[t_2]h.$$

I point out that this can be looked at as a dot product of terms coming from the tables for computing each of $[t_0, t_1, t_2]g$ and $[t_0, t_1, t_2]h$ by going along the top of the g table and down from right to left along the bottom of the h table. They have a homework exercise to do this for some rational functions.

The proof is really cool! We start with the question: If $p(t)$ is the interpolating polynomial to f at t_0, t_1, t_2 and $q(t)$ and $r(t)$ are the corresponding ones for g and h, can't we just use qr for p? We check the interpolation conditions and see that qr matchs f nicely at the t_i's. The problem is that it has the wrong degree, since it is only guaranteed to be in P_4, not P_2. The cool part is that we can actually find p hiding inside the product qr if we multiply it out very carefully! So, we write the Newton forms for q and r as follows:

$$q(t) = [t_0]g + [t_0, t_1]g(t - t_0) + [t_0, t_1, t_2]g(t - t_0)(t - t_1),$$
$$r(t) = [t_2]h + [t_1, t_2]h(t - t_2) + [t_0, t_1, t_2]h(t - t_2)(t - t_1).$$

Note: We can write r backwards like this since the order of the values t_i does not matter, as we noted before when proving the Newton form. Next we notice that the terms of degree 3 or higher in the product $q(t)r(t)$ all vanish at each of the t_i. So the terms of degree at most 2 form a polynomial in P_2 which meets all the conditions for p. By the uniquness of p, this is indeed p. Now we simply extract the coefficient of t^2 and get the Leibniz formula.

A student asked how this works for equal t_i's. Good question!