Lecture 16

Main Points:

- Derivatives of Bezier Curves
- Non-degnerate quadratic Bezier Curves

Derivatives of parametric polynomial curves

The derivative of a parametric curve

$$\gamma(t) = (x(t), y(t))$$

is simply:

$$\gamma'(t) = (x'(t), y'(t)).$$

The curve is called differentiable at t_0 if this derivative exists at t_0. The curve is called *smooth* at t_0 if this derivative exists and is nonzero at t_0.

The derivative can be interpreted as a velocity vector in the direction of increasing t if the curve $\gamma(t)$ is traversed by a particle for time t.

The tangent line at a point $\gamma(t_0) = (x_0, y_0)$ on a differentiable curve $\gamma(t)$ is defined to be the line through (x_0, y_0) with direction vector $\gamma'(t_0)$.

Cumulative form for Bezier curves

A Bezier curve can be written in BB-form:

$$\gamma(t) = \sum_{i=0}^{d} B_i^d(t) P_i,$$

where $B_i^d(t)$, for $i = 0, \ldots, d$ are the Bernstein polynomials. We can also use the cumulative Bernstein polynomials to write $\gamma(t)$ in a new form which is convenient for derivatives. First, we recall the definition of the cumulative Bernstein polynomials:

$$C_i^d(t) = \sum_{j=i}^{d} B_j^d(t),$$

which are also defined for $i = 0, \ldots, d$. Since each summation for $C_i^d(t)$ starts with $B_i^d(t)$ and continues to add the higher indexed Bernstein polynomials, we can write:

$$C_i^d(t) = B_i^d(t) + \sum_{j=i+1}^{d} B_j^d(t) = B_i^d(t) + C_{i+1}^d(t),$$

except for the case $i = d$, in which case $C_i^d(t) = B_i^d(t)$. This then leads to:

$$\sum_{i=0}^{d} C_i^d(t) = \sum_{i=0}^{d} B_i^d(t) + \sum_{i=0}^{d-1} C_{i+1}^d(t).$$

We can then write the sum with control points:

$$\begin{aligned}\sum_{i=0}^{d} C_i^d(t) P_i &= \sum_{i=0}^{d} B_i^d(t) P_i + \sum_{i=0}^{d-1} C_{i+1}^d(t) P_i \\ &= \sum_{i=0}^{d} B_i^d(t) P_i + \sum_{i=1}^{d} C_i^d(t) P_{i-1},\end{aligned}$$

which allows us to solve for:

$$\begin{aligned}
\gamma(t) &= \sum_{i=0}^{d} B_i^d(t) P_i \\
&= \sum_{i=0}^{d} C_i^d(t) P_i - \sum_{i=1}^{d} C_i^d(t) P_{i-1} \\
&= C_0^d(t) P_0 + \sum_{i=1}^{d} C_i^d(t) \left(P_i - P_{i-1} \right) \\
&= P_0 + \sum_{i=1}^{d} C_i^d(t) \mathbf{v}_i,
\end{aligned}$$

where $\mathbf{v}_i$ is the vector from P_{i-1} to P_i.

Derivatives of Bezier curves with Cumulative form

Recall the derivative of the cumulative Bernstein polynomials:

$$\frac{d}{dt} C_i^d(t) = d B_{i-1}^{d-1}(t),$$

for $i = 0, \ldots, d$. This can be used to get the derivative of the cumulative form of the Bezier curve:

$$\begin{aligned}
\gamma'(t) &= \frac{d}{dt} \left[P_0 + \sum_{i=1}^{d} C_i^d(t) \mathbf{v}_i \right] \\
&= \sum_{i=1}^{d} \frac{d}{dt} \left[C_i^d(t) \right] \mathbf{v}_i \\
&= \sum_{i=1}^{d} d B_{i-1}^{d-1}(t) \mathbf{v}_i \\
&= d \sum_{i=0}^{d-1} B_i^{d-1}(t) \mathbf{v}_{i+1}.
\end{aligned}$$

Examples:

- From the above formula we can deduce:

  $$\gamma'(0) = d\mathbf{v}_1.$$

  For example, $P_0 = (0,0)$, $P_1 = (2,3)$, and $P_2 = (5,7)$, then we can say that the quadratic Bezier curve with these control points must have tangent vector at $t = 0$ given by two times the vector between the first two control points:

  $$\gamma'(0) = 2\mathbf{v}_1 = 2\left(P_1 - P_0\right) = 2\left((2,3) - (0,0)\right) = (4,6).$$

- From the above formula we can also deduce:

  $$\gamma'(1) = d\mathbf{v}_2.$$

  Again, if $P_0 = (0,0)$, $P_1 = (2,3)$, and $P_2 = (5,7)$, then we can say that the quadratic Bezier curve with these control points must have tangent vector at $t = 1$ given by two times the vector between the last two control points:

  $$\gamma'(1) = 2\mathbf{v}_2 = 2\left(P_2 - P_1\right) = 2\left((5,7) - (2,3)\right) = (3,4).$$

Degenerate and non-degenerate quadratic Bezier curves

A quadratic Bezier curve $\gamma(t)$ can be define by its BB-form with 3 control points P_0, P_1, and P_2. If all of these control points are collinear, then we call $\gamma(t)$ a degenerate quadratic Bezier curve. In this case all of the points of $\gamma(t)$ lie on the same line. This must be the case since any point $\gamma(t)$ can be computed by nested linear interpolation, starting with the control points, and thus can never be off the line. If the control points are not all collinear, then we call the curve non-degenerate.

Examples:

- For example, if we define:

$$\gamma(t) = (1-t)^2(2,-1) + 2(1-t)t(0,1) + t^2(-1,2),$$

  then since the control points all lie on the line $y = -x + 1$, we must have all the points of $\gamma(t)$ on this line.

- We can also reverse this process and define a quadratic parametric curve which clearly must have all its points on a line. For example:

$$\gamma(t) = (t^2 - 2t + 3, t^2 - 2t)$$

  satisfies the linear relationship $y = x - 3$ since:

$$x - 3 = (t^2 - 2t + 3) - 3 = t^2 - 2t = y.$$

  So if we write this curve in BB-form with three control points, it must be the case that these control points are collinear. We can find the control points with the polar form:

$$F[u_1, u_2] = (3,0) + (-2,-2)\frac{1}{2}[u_1 + u_2] + (1,1)u_1u_2.$$

  This yields:

$$P_0 = F[0,0] = (3,0), \qquad P_1 = F[0,1] = (1,-2), \qquad P_2 = F[1,1] = (2,-1),$$

  which are all on the line $y = x - 3$.

Every non-degenerate quadratic Bezier curve has all points lying on a parabola

A quadratic Bezier curve is one form of a quadratic polynomial parametric curve. In the standard basis, such a curve could be defined as:

$$\gamma(t) = (x, y) = (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2).$$

Suppose that $\gamma(t)$ is non-degenerate. In order to show that such curves must have all points lying on a parabola, we will first treat the case where at least one of a_2 or b_2 is zero.

Suppose $a_2 = 0$. Then we can solve for t in terms of x:

$$t = \frac{1}{a_1}(x - a_0).$$

Note: We can divide by a_1 since we cannot have both a_2 and a_1 equal to zero, otherwise the x-coordinate of $\gamma(t)$ would be constant and all its points would lie on a vertical line, but we are assuming that $\gamma(t)$ is non-degnerate. Substituting, we have:

$$y = b_0 + b_1 t + b_2 t^2 = b_0 + b_1 \frac{1}{a_1}(x - a_0) + b_2 \frac{1}{a_1^2}(x - a_0)^2.$$

Collecting terms and completing the square, we can then write such a quadratic as:

$$y = a(x - b) + c^2,$$

which is a standard form for a parabola with vertex at (b, c) and axis of symmetry parallel to the y-axis.

The case where $b_2 = 0$ produces a standard form for a parabola with axis of symmetry parallel to the x-axis.

Now suppose that both a_2 and b_2 are nonzero. We will show that there is a linear change of coordinates which is in fact simply a rotation of coordinates, which represents $\gamma(t)$ as a parabola.

We can represent any linear change of coordinates as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words:

$$x' = a_{1,1}x + a_{1,2}y = a_{1,1}(a_0 + a_1 t + a_2 t^2) + a_{1,2}(b_0 + b_1 t + b_2 t^2),$$

and

$$y' = a_{2,1}x + a_{2,2}y = a_{2,1}(a_0 + a_1 t + a_2 t^2) + a_{2,2}(b_0 + b_1 t + b_2 t^2).$$

In order for such a coordinate system to represent $\gamma(t)$ as a parabola, we would need to have the t^2 coefficient in x' or in y' be equal to zero. For instance, we can force this in x' if we take

$$a_{1,1} = b_2, \qquad \text{and} \qquad a_{1,2} = -a_2.$$

The transformation matrix then becomes:

$$A = \begin{pmatrix} b_2 & -a_2 \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

But we wanted to produce a rotation, so we will need to have also $a_{2,1} = a_2$, and $a_{2,2} = b_2$. Additionally, we need to have determinant one, which can be achieved by multiplying the matrix by the constant

$$\frac{1}{\delta} = \frac{1}{\sqrt{a_2^2 + b_2^2}}$$

producing:

$$A = \frac{1}{\delta} \begin{pmatrix} b_2 & -a_2 \\ a_2 & b_2 \end{pmatrix}.$$

This is a rotation matrix and thus the equation for $\gamma(t)$ in the new coordinates x' and y' will be a parabola. Since a rotated parabola is still a parabola, we must conclude that the original graph is also a parabola.

Implicit quadratic equations of conics and the discriminant

The general quadratic equation of a conic in x and y is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The discriminant of this equation is:

$$\Delta = B^2 - 4AC.$$

The graph of the conic is called non-degenerate if it is an ellipse, parabola, or hyperbola. Other cases, such as a single point, a line, or a pair of lines, are called degenerate. If the graph is non-degenerate, the type of the graph can be determined from the following:

$$\begin{array}{rcl} \Delta < 0 & \longleftrightarrow & \mathit{Ellipse} \\ \Delta = 0 & \longleftrightarrow & \mathit{Parabola} \\ \Delta > 0 & \longleftrightarrow & \mathit{Hyperbola} \end{array}$$

Examples:

- $x^2 + y^2 = 1$ has $A = C = 1$ and $B = 0$, so $\Delta = B^2 - 4AC = -1$, an ellipse (circle).
- $y = x^2$ has $A = -1$ and $B = C = 0$, so $\Delta = B^2 - 4AC = 0$, a parabola.
- $xy = 1$ has $A = C = 0$ and $B = 1$, so $\Delta = B^2 - 4AC = 1$, a hyperbola.

Five point construction of conics

In order to construct the implicit equation for a quadratic Bezier curve, we will use a geometric technique which starts with the five point construction. This construction allows us to find the equation of a conic, or quadratic polynomial in x and y, which passes through any collection of five points.

Suppose we are given 5 points P_0, P_1, P_2, P_3 and P_4 in the plane. We would like to find an equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which satisfies all five points.

We start by writing two pairs of lines through the first 4 points. Suppose the first pair of lines is

$$L_{0,1}(x,y) = 0 \qquad \text{and} \qquad L_{2,3}(x,y) = 0$$

where $L_{0,1}$ passes through P_0 and P_1, and $L_{2,3}$ passes through P_2 and P_3; and the second pair is

$$L_{0,2}(x,y) = 0 \qquad \text{and} \qquad L_{1,3}(x,y) = 0$$

where $L_{0,2}$ passes through P_0 and P_2, and $L_{1,3}$ passes through P_1 and P_3.

Next, we write the quadratic equation

$$f_c(x,y) = L_{0,1}(x,y)L_{2,3}(x,y) + L_{0,2}(x,y)L_{1,3}(x,y) = 0,$$

which represents a family of conics, each of which passes through the four points P_0, P_1, P_2, and P_3.

The final step is to solve for c after plugging in the coordinates of P_4 into the equation for $f_c(x,y)$. This will guarantee that $f_c(x,y)$ also passes through the fifth point.

Examples:

- Find the conic which passes through the points: $P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (2,1)$, $P_3 = (0,1)$, and $P_4 = (1,4)$. Note: the points are chosen to be suggestive of the shape of an ellipse, and are indeed inconsistent with the shape of a parabola or hyperbola. So we expect that the equation will have negative discriminant.

  We form the lines:

  $$L_{0,1}(x,y) = y = 0, \qquad \text{and} \qquad L_{2,3}(x,y) = y - 1 = 0,$$
  $$L_{0,2}(x,y) = x - 2y = 0, \qquad \text{and} \qquad L_{2,3}(x,y) = x + y - 1 = 0.$$

  Next, we form $f_c(x,y)$:

  $$\begin{aligned} f_c(x,y) &= L_{0,1}(x,y)L_{2,3}(x,y) + L_{0,2}(x,y)L_{1,3}(x,y) = 0 \\ &= y(y-1) + c(x-2y)(x+y-1) \\ &= 0. \end{aligned}$$

  Now we insert the coordinates of $P_4 = (1,4)$:

  $$\begin{aligned} f_c(1,4) &= 4(4-1) + c(1 - 2 \cdot 4)(1 + 4 - 1) \\ &= 12 - 28c \\ &= 0, \end{aligned}$$

  which means that $c = \frac{12}{28} = \frac{3}{7}$, and the equation of the desired concic is:

  $$f_c(x,y) = y(y-1) + \frac{3}{7}(x-2y)(x+y-1) = 0,$$

which can also be written as:

$$7y(y-1)+3(x-2y)(x+y-1)=0.$$

Expanding and simplifying, we get:

$$3x^2-3xy+y^2-3x-y=0.$$

Finally, we check the discriminant and get:

$$\Delta = B^2-4AC = 9-12 = -3 < 0$$

and we know that we indeed have an ellipse.

Tangent construction of conics

Suppose that in the five point construction we let the point P_1 approach P_0 along the line between them until they finally meet at the point P_0. If we could watch the conics smoothly deform as we perform this transition, we would see that the limit as P_1 approaches P_0 is in fact the conic which has tangent line at P_0 given by the initial line between P_0 and P_1.

We can do the same with the points P_2 and P_3, allowing P_3 to approach P_2, and obtaining a conic in the limit which has tangent line at P_2 given by the initial line through P_2 and P_3. In this process we can also note that the two lines $L_{0,2}$ and $L_{2,3}$ have become the same line, which we call simply L. We also relabel the lines $L_{0,2}$ to L_0 and $L_{2,3}$ to L_2, since these are now the tangent lines at P_0 and P_2. We then have:

$$f_c(x,y)=L_0(x,y)L_2(x,y)+cL(x,y)^2=0.$$

We can solve for c just as we did in the five point construction, by inserting P_4 into the equation. The resulting equation will then pass through P_0, P_2, and P_4, and have tangent lines L_0 at P_0 and L_2 at P_2.

The above discussion gives an intuitive idea of how these constructions work. A full verification requires techniques in algebraic geometry, which we will not pursue here.

Implicit form of a quadratic Bezier curve

We can apply the above tangent construction to the case of a quadratic Bezier curve $\gamma(t)$. This makes it possible to start with the three control points P_0, P_1, and P_2 and obtain from these the implicit equation $f(x,y)=0$ for the parabola which represents the curve $\gamma(t)$.

If P_0, P_1, and P_2 are collinear, we can easily find a linear equation $f(x,y)=0$ that represents this degenerate Bezier curve, so we will now assume that they are not collinear, and thus the Bezier curve has an implicit form which is a parabola.

Since we know that the tangent vectors to $\gamma(t)$ at $t=0$ and $t=1$ are parallel to the the line L_0 which passes through P_0 to P_1, and the line L_2 which passes through P_1 to P_2, we can use those lines as the tangent lines at P_0 and P_2. We can also find the line between P_0 and P_2 and call it L. Then we can write the equation:

$$f_c(x,y)=L_0(x,y)L_2(x,y)+cL(x,y)^2=0.$$

To solve for c, we could use a point like $\gamma(\frac{1}{2})$, which we know must be on the curve, which yields the implicit form of the curve $\gamma(t)$.