Lecture 17

Main Points:

- Non-degenerate quadratic Bezier curves as parabolas
- Five point construction of conics

Every non-degenerate quadratic Bezier curve has all points lying on a parabola

A quadratic Bezier curve is one form of a quadratic polynomial parametric curve. In the standard basis, such a curve could be defined as:

$$\gamma(t) = (x, y) = (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2).$$

Suppose that $\gamma(t)$ is non-degenerate. In order to show that such curves must have all points lying on a parabola, we will first treat the case where at least one of a_2 or b_2 is zero.

Suppose $a_2 = 0$. Then we can solve for t in terms of x:

$$t = \frac{1}{a_1}(x - a_0).$$

Note: We can divide by a_1 since we cannot have both a_2 and a_1 equal to zero, otherwise the x-coordinate of $\gamma(t)$ would be constant and all its points would lie on a vertical line, but we are assuming that $\gamma(t)$ is non-degenerate. Substituting, we have:

$$y = b_0 + b_1 t + b_2 t^2 = b_0 + b_1 \frac{1}{a_1}(x - a_0) + b_2 \frac{1}{a_1^2}(x - a_0)^2.$$

Collecting terms and completing the square, we can then write such a quadratic as:

$$y = a(x - b) + c^2,$$

which is a standard form for a parabola with vertex at (b, c) and axis of symmetry parallel to the y-axis.

The case where $b_2 = 0$ produces a standard form for a parabola with axis of symmetry parallel to the x-axis.

Now suppose that both a_2 and b_2 are nonzero. We will show that there is a linear change of coordinates which is in fact simply a rotation of coordinates, which represents $\gamma(t)$ as a parabola.

We can represent any linear change of coordinates as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words:

$$x' = a_{1,1}x + a_{1,2}y = a_{1,1}(a_0 + a_1 t + a_2 t^2) + a_{1,2}(b_0 + b_1 t + b_2 t^2),$$

and

$$y' = a_{2,1}x + a_{2,2}y = a_{2,1}(a_0 + a_1 t + a_2 t^2) + a_{2,2}(b_0 + b_1 t + b_2 t^2).$$

In order for such a coordinate system to represent $\gamma(t)$ as a parabola, we would need to have the t^2 coefficient in x' or in y' be equal to zero. For instance, we can force this in x' if we take

$$a_{1,1} = b_2, \qquad \text{and} \qquad a_{1,2} = -a_2.$$

The transformation matrix then becomes:

$$A = \begin{pmatrix} b_2 & -a_2 \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

But we wanted to produce a rotation, so we will need to have also $a_{2,1} = a_2$, and $a_{2,2} = b_2$. Additionally, we need to have determinant one, which can be achieved by multiplying the matrix by the constant

$$\frac{1}{\delta} = \frac{1}{\sqrt{a_2^2 + b_2^2}}$$

producing:

$$A = \frac{1}{\delta}\begin{pmatrix} b_2 & -a_2 \\ a_2 & b_2 \end{pmatrix}.$$

This is a rotation matrix and thus the equation for $\gamma(t)$ in the new coordinates x' and y' will be a parabola. Since a rotated parabola is still a parabola, we must conclude that the original graph is also a parabola.

Five point construction of conics

In order to construct the implicit equation for a quadratic Bezier curve, we will use a geometric technique which starts with the five point construction. This construction allows us to find the equation of a conic, or quadratic polynomial in x and y, which passes through any collection of five points.

Suppose we are given 5 points P_0, P_1, P_2, P_3 and P_4 in the plane. We would like to find an equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which satisfies all five points.

We start by writing two pairs of lines through the first 4 points. Suppose the first pair of lines is

$$L_{0,1}(x, y) = 0 \quad \text{and} \quad L_{2,3}(x, y) = 0$$

where $L_{0,1}$ passes through P_0 and P_1, and $L_{2,3}$ passes through P_2 and P_3; and the second pair is

$$L_{0,2}(x, y) = 0 \quad \text{and} \quad L_{1,3}(x, y) = 0$$

where $L_{0,2}$ passes through P_0 and P_2, and $L_{1,3}$ passes through P_1 and P_3.

Next, we write the quadratic equation

$$f_c(x, y) = L_{0,1}(x, y)L_{2,3}(x, y) + L_{0,2}(x, y)L_{1,3}(x, y) = 0,$$

which represents a family of conics, each of which passes through the four points P_0, P_1, P_2, and P_3.

The final step is to solve for c after plugging in the coordinates of P_4 into the equation for $f_c(x, y)$. This will guarantee that $f_c(x, y)$ also passes through the fifth point.

Examples:

- Find the conic which passes through the points: $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (2, 1)$, $P_3 = (0, 1)$, and $P_4 = (1, 4)$. Note: the points are chosen to be suggestive of the shape of an ellipse, and are indeed inconsistent with the shape of a parabola or hyperbola. So we expect that the equation will have negative discriminant.

  We form the lines:

  $$L_{0,1}(x, y) = y = 0, \quad \text{and} \quad L_{2,3}(x, y) = y - 1 = 0,$$
  $$L_{0,2}(x, y) = x - 2y = 0, \quad \text{and} \quad L_{1,3}(x, y) = x + y - 1 = 0.$$

  Next, we form $f_c(x, y)$:

  $$\begin{aligned} f_c(x, y) &= L_{0,1}(x, y)L_{2,3}(x, y) + L_{0,2}(x, y)L_{1,3}(x, y) = 0 \\ &= y(y - 1) + c(x - 2y)(x + y - 1) \\ &= 0. \end{aligned}$$

Now we insert the coordinates of $P_4 = (1, 4)$:

$$\begin{aligned} f_c(1,4) &= 4(4-1) + c(1-2\cdot 4)(1+4-1) \\ &= 12 - 28c \\ &= 0, \end{aligned}$$

which means that $c = \frac{12}{28} = \frac{3}{7}$, and the equation of the desired concic is:

$$f_c(x, y) = y(y-1) + \frac{3}{7}(x-2y)(x+y-1) = 0,$$

which can also be written as:

$$7y(y-1) + 3(x-2y)(x+y-1) = 0.$$

Expanding and simplifying, we get:

$$3x^2 - 3xy + y^2 - 3x - y = 0.$$

Finally, we check the discriminant and get:

$$\Delta = B^2 - 4AC = 9 - 12 = -3 < 0$$

and we know that we indeed have an ellipse.

- If we choose five points in certain configurations, we expect to get a hyperbola. For example, suppose $P_0 = (0, 3)$, $P_1 = (3, 0)$, $P_2 = (0, -3)$ and $P_3 = (-3, 0)$. If we now choose $P_4 = (0, 0)$, then we clearly cannot have a parabola nor an ellipse passing through these five points, so we expect to get a hyperbola. In this case, however, there is already a degenerate conic given by the product of the lines $x = 0$ and $y = 0$ which passes through all five points. So, we choose a slightly less symmetric fifth point: $P_4 = (1, 1)$. It seems that in this case, we should get a hyperbola which has one branch passing through P_0, P_1 and P_4, and another symmetric branch passing through P_2, P_3, and $(-1, -1)$.

  We form the lines:

$$L_{0,1}(x, y) = x + y - 3 = 0, \quad \text{and} \quad L_{2,3}(x, y) = x + y + 3 = 0,$$
$$L_{0,2}(x, y) = x = 0, \quad \text{and} \quad L_{1,3}(x, y) = y = 0.$$

  Next, we form $f_c(x, y)$:

$$\begin{aligned} f_c(x,y) &= L_{0,1}(x,y)L_{2,3}(x,y) + L_{0,2}(x,y)L_{1,3}(x,y) = 0 \\ &= (x+y-3)(x+y+3) + cxy \\ &= 0. \end{aligned}$$

  Now we insert the coordinates of $P_4 = (1, 1)$:

$$\begin{aligned} f_c(1,1) &= (-1)(5) + c \cdot 1 \\ &= -5 + c \\ &= 0, \end{aligned}$$

  which means that $c = 5$ and the equation of the desired concic is:

$$f_c(x, y) = (x+y-3)(x+y+3) + 5xy.$$

  Expanding and simplifying, we get:

$$x^2 + 7xy + y^2 - 9 = 0.$$

  Finally, we check the discriminant and get:

$$\Delta = B^2 - 4AC = 49 - 4 = 45 > 0$$

  and we know that we indeed have a hyperbola. We also see that the hyperbola passes through $(-1, -1)$ as expected.