Lecture 19

Main Points:

- Further examples of quadratic Bezier Curves

More Examples:

- Assume that a Bezier curve has implicit form $y = x^2$, and we know that $P_0 = (-2, 4)$ and $P_1 = (-\frac{1}{2}, -2)$. We will find the third control point P_2.

  Note: Once we specify $P_0 = (-2, 4)$, we cannot choose P_1 arbitrarily since it must lie on the tangent line to $y = x^2$ at the point $(-2, 4)$. Since the derivative $y' = 2x$ has value -4 at $x = -2$, we see that this tangent line has equation $y = -4x - 4$ and indeed $(-\frac{1}{2}, -2)$ is on this line.

  Next, we need to find a point P_2, on $y = x^2$, which must have the property that the tangent line at P_2 passes through P_1. Suppose that the point P_2 is written:

  $$P_2 = (a, b) = (a, a^2).$$

  Then the tangent line at $P_2 = (a, a^2)$ has slope $2a$ and can be written:

  $$y = 2a(x - a) + a^2.$$

  Now we can plug in $P_1 = (-\frac{1}{2}, -2)$ to this line to solve for a:

  $$-2 = 2a(-\frac{1}{2} - a) + a^2,$$

  or

  $$-2 = -a - a^2,$$

  or

  $$a^2 + a - 2 = (a + 2)(a - 1) = 0.$$

  We expect to find two solutions, since we already know that the tangent line at $P_0 = (-2, 4)$ does pass through P_1. This corresponds to $a = -2$ in the above quadratic equation. The new solution is $a = 1$, which gives us the point

  $$P_2 = (a, a^2) = (1, 1).$$

  Once again, we can perform a consistency check by computing the point $\gamma(\frac{1}{2})$ with these control points, which should be a point on the curve $y = x^2$.

  $$\begin{array}{ccccc} P_0 = (-2, 4) & & & & \\ & & P_0^1 = (-\frac{5}{4}, 1) & & \\ P_1 = (-\frac{1}{2}, -2) & & & & P_0^2 = (-\frac{1}{2}, \frac{1}{4}) = \gamma(\frac{1}{2}) \\ & & P_1^1 = (\frac{1}{4}, -\frac{1}{2}) & & \\ P_2 = (1, 1) & & & & \end{array}$$

  and indeed $\gamma(\frac{1}{2}) = (-\frac{1}{2}, \frac{1}{4})$ is a point on $y = x^2$.

- What happens if we have collinear control points in the tangent construction? Suppose $P_0 = (0, 0)$, $P_1 = (1, 0)$ and $P_2 = (3, 0)$.

  In this case, since all the control points are on the x-axis, we have

  $$L_0(x, y) = y = 0, \qquad L_2(x, y) = y = 0, \qquad \text{and} \qquad L(x, y) = y = 0,$$

which gives:

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = y^2 + cy^2 = 0.$$

Now, since any other point on the Bezier curve with these control points can be obtained by nested linear interpolation, it must also be on the same line. So any constant c will do, and we have simply:

$$f_c(x,y) = y^2 = 0.$$

This is just a "double line", which is technically a quadratic equation, whose graph is however simply a line.

The Bezier curve with these control points:

$$\begin{aligned}
\gamma(t) &= (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \\
&= (1-t)^2(0,0) + 2(1-t)t(1,0) + t^2(3,0) \\
&= (2(1-t)t + 3t^2, 0) \\
&= (2t + t^2, 0)
\end{aligned}$$

gives a parametrization which lies on the line $y = 0$, however it does not travel the line in a typical way. Instead, it proceeds in one direction, stops and turns around, and then proceeds in the opposite direction. To see where it stops, we can consider the derivative:

$$\gamma'(t) = 2\left[(1-t)\mathbf{v}_1 + t\mathbf{v}_2\right] = 2\left[(1-t)(1,0) + t(2,0)\right] = 2(1+t,0).$$

A particle moving with this velocity vector will stop when the vector is zero, and we see that $\gamma'(t) = (0,0)$ exactly for $t = -1$. At $t = 0$ it is at $P_0 = (0,0)$, with velocity vector $(2,0)$, and at $t = 1$ it is at $P_2 = (2,0)$, with velocity vector $(4,0)$.

So we see that the particle must come from $+\infty$ on the x-axis, as t comes from $-\infty$, then pass through P_2 and P_0 for some negative values of t, then when $t = -1$ it reaches the point $(-1,0)$, turns around, and heads back towards $+\infty$ on the x-axis. We can think of this trajectory as a "squashed parabola", which has been flattened so that its vertex is now at $(-1,0)$.

- Find the intersection points of two Bezier curves. Let $\gamma(t)$ have control points $P_0 = (0,2)$, $P_1 = (0,0)$, and $P_2 = (2,0)$. We found the implicit equation to be:

  $$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0.$$

  Now let $\alpha(t)$ have control points $Q_0 = (0,2)$, $Q_1 = (-1,1)$, and $Q_2 = (2,0)$. Then clearly $\gamma(t)$ and $\alpha(t)$ have at least the two points $P_0 = Q_0$ and $P_2 = Q_2$ in common. From the graphs we can see that they also must have another point in common, with coordinates between 0 and 1.

  First, we need the implicit form for $\alpha(t)$. We find the linear equations:

  $$L_0(x,y) = x - y + 2 = 0, \qquad L_2(x,y) = x + 3y - 2 = 0, \qquad \text{and} \qquad L(x,y) = x + y - 2 = 0,$$

  which gives:

  $$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = (x - y + 2)(x + 3y - 2) + c(x + y - 2)^2 = 0.$$

  Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

  $$\begin{array}{ccccc}
  P_0 = (0,2) & & & & \\
  & & P_0^1 = (-\frac{1}{2}, \frac{3}{2}) & & \\
  P_1 = (-1,1) & & & & P_0^2 = (0,1) = \alpha(\frac{1}{2}) \\
  & & P_1^1 = (\frac{1}{2}, \frac{1}{2}) & & \\
  P_2 = (2,0) & & & &
  \end{array}$$

Finally, we plug in $(0,1)$ to $f_c(x,y) = 0$ and solve for c:

$$\begin{aligned} f_c(0,1) &= (0-1+2)(0+3-2) + c(0+1-2)^2 \\ &= 1+c \\ &= 0, \end{aligned}$$

which means that $c = -1$, and the equation for $f_c = f$ is:

$$f(x,y) = (x-y+2)(x+3y-2) - (x+y-2)^2 = 0.$$

This is equivalent to:

$$4x - 4y^2 + 12y - 8 = 0,$$

or

$$x = y^2 - 3y + 2 = (y - \frac{3}{2})^2 - \frac{1}{4}.$$

Now to find the intersection points, we substitute $x = y^2 - 3y + 2$ into the equation of the first parabola

$$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0$$

to get:

$$(y^2 - 3y + 2)^2 + y^2 - 2(y^2 - 3y + 2)y - 4(y^2 - 3y + 2) - 4y + 4 = 0$$

and simplifying, we have:

$$y^4 - 8y^3 + 16y^2 - 8y = 0.$$

Since we know that these two parabolas do intersect at $P_0 = (0,2)$ and $P_2 = (2,0)$, we know that the y-coordinates of these points must be solutions of this fourth degree polynomial. Indeed, we can factor out $y(y-2)$ to get:

$$y(y-2)(y^2 - 6y + 4) = 0$$

which also has the solutions:

$$y = 3 \pm \sqrt{5},$$

which gives the approximate points of intersection:

$$(13.7, 5.2) \qquad \text{and} \qquad (0.29, 0.76).$$

Resultants

To intersect to general quadratic equations, we can use the resultant. Suppose that

$$f(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2,$$

and

$$g(x,y) = b_0(x) + b_1(x)y + b_2(x)y^2.$$

Then the resultant is defined as a determinant:

$$R(x) = \begin{vmatrix} a_0(x) & 0 & b_0(x) & 0 \\ a_1(x) & a_0(x) & b_1(x) & b_0(x) \\ a_2(x) & a_1(x) & b_2(x) & b_1(x) \\ 0 & a_2(x) & 0 & b_2(x) \end{vmatrix}.$$

The x-coordinates of the intersection points of $f(x,y) = 0$ and $g(x,y) = 0$ are zeros of the resultant $R(x)$. Similarly, we can define a resultant $R(y)$, whose zeros are the y-coordinates of the intersection points.