Lecture 2

Main Points:

- Finish Overview of Project Part II - Bezier Curves, BB-form, Midpoint Subdivision
- Vector Spaces of Polynomials: Shifted, Vandermonde, and Top-down bases.
- Bernstein Polynomials and Bernstein basis.

Midpoint Subdivision (Refer to Project Part II subpart 3)

The third sub-part of Project II, Midpoint Subdivision, differs from the other two in an important way: the points which are generated are not necessarily on the curve. However, they are always either on the curve or on a tangent line to the curve. Since the number of points increases, this guarantees that the line segments converge to the actual curve. Another way in which it differs is that it produces successive approximations to the curve recursively, rather than producing a list of points to be connected.

Polynomial vector spaces

P_d is the vector space of polynomials of degree at most d. (In this course we will typically use the variable t for polynomials.)

Examples:

- The vector space P_3 consists of polynomials of degree at most 3 such as $2 - 4t + 8t^2 - 6t^3$, $4t - t^3$, $1 + 6t$, or even constant polynomials (degree zero) such as 3, -2, or 0, etc.

Addition and scalar multiplication in P_d are done in the usual way for polynomials, which is the usual addition of polynomials and multiplication by real numbers that is familiar from algebra.

The **standard basis** of P_d is: $\{1, t, t^2, \ldots, t^d\}$. The **dimension** of P_d is $d + 1$.

The **coordinate vector** of a polynomial with respect to the standard basis is a column vector of coefficients. The default order for the coefficients is with increasing degree corresponding to order in the column vector from top to bottom.

Examples:

- The polynomial $2 - 3t + 4t^2$ in P_2 has coordinate vector with respect to the standard basis given by: $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Shifted bases are formed with one constant c as: $\{1, t - c, (t - c)^2, \ldots, (t - c)^d\}$.

Examples:

- A shifted basis of P_2 is: $\{1, t-3, (t-3)^2\}$. The polynomial $3-2(t-3)+2(t-3)^2$ has coorinate vector with respect to this basis: $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$. Since $3-2(t-3)+2(t-3)^2$ is equal (after multiplying out) to: $27-14t+2t^2$, we also see that this polynomial has coordinate vector with respect to the standard basis: $\begin{pmatrix} 27 \\ -14 \\ 2 \end{pmatrix}$. It is important to understand that these two coordinate vectors represent exactly the same polynomial, but the coordinates are different because they represent the coefficients with respect to different bases.

Vandermonde bases are formed with $d+1$ constants $t_0, t_1, \ldots, t_d$ as: $\{(t-t_0)^d, (t-t_1)^d, \ldots, (t-t_d)^d\}$.

Examples:

- $\{(t-1)^2, (t-3)^2, (t-9)^2\}$ is a Vandermond basis of P_2, with $t_0=1$, $t_1=3$, $t_2=9$.
- $\{t^3, (t+1)^3, (t-4)^3, (t-5)^3\}$ is a Vandermond basis of P_3, with $t_0=0$, $t_1=-1$, $t_2=4$, and $t_3=5$.

Top-down bases (intuitive description)

A **top-down basis** of P_d is a set of $d+1$ polynomials which can be either i) Vandermonde, or ii) Shifted, or iii) in between. The third case can be thought of as follows: Take several shifted bases and list them in columns from highest degree on top, to lowest degree on bottom. From these columns, choose any number of polynomials from the top downward, without skipping any, so that the total number of polynomials adds up to $d+1$. This is a top-down basis. If you only use the top elements of each column, you get a Vandermonde basis. If you go all the way down one column, you get a shifted basis. The in between cases give you pieces of shifted bases collected together, with the requirement that the pieces start with highest degree and work down (top-down).

Examples:

- $\{(t-1)^2, t-2, (t-3)^2\}$ is a basis which is not a top-down basis of P_2.
- Exactly 10 top-down bases can be formed from following the grid of polynomials:

$$\begin{array}{ccc} (t-1)^2 & (t-2)^2 & (t-3)^2 \\ t-1 & t-2 & t-3 \\ 1 & 1 & 1 \end{array}$$

  One of them is also Vandermonde: $\{(t-1)^2, (t-2)^2, (t-3)^2\}$. Each column makes up one shifted basis. The other cases are of the type: $\{(t-1)^2, t-1, (t-2)^2\}$, which take two from the top of one stack and one from another. (Check that there are ten in all.)

Top-down bases (detailed technical description)

The **top-down bases** are the sets obtained from a grid of polynomials in the following way: Choose distinct real numbers $t_0 < t_1 < \cdots < t_r$. The grid $G(t_0, t_1, \ldots, t_r)$ consists of rows $(t-t_0)^{d-i}, (t-t_1)^{d-i}, \ldots, (t-t_r)^{d-i}$ for $i = 0, \ldots, d$. (Another way to define this grid is as a $(d+1)\times(r+1)$ matrix with (i,j) entry given as: $(t-t_{j-1})^{d-i+1}$.) A **top-down set** S is a set of at most $d+1$ polynomials taken from this grid by choosing from the top of some subset of the $r+1$ columns and working down. Any number of polynomials can be taken from each column, up to a maximum of $d+1$. Thus, if $(t-t_j)^{d-i}$ is a member of S, then $(t-t_j)^d, (t-t_j)^{d-1}, \ldots, (t-t_j)^{d-i+1}$ must also be members of S. If a set of $d+1$ polynomials is chosen in this way, it is called a top-down basis of P_d. We can also describe these sets without the visual aid of the grid as follows: Choose distinct real numbers $t_0 < t_1 < \cdots < t_r$, and indices $m_0, m_1, \ldots, m_r$ with $0 \le m_i \le d+1$, and the sum $\sum_{i=0}^{r} m_i = d+1$. Then the top-down set associated to this data is the union of the sets

$$\{(t-t_i)^d, (t-t_i)^{d-1}, \ldots, (t-t_i)^{d-m_i+1}\}, \quad i = 0, \ldots, r$$

where if $m_i = 0$ then the set is empty.

Change of basis

To change between one basis and another, we need a change of basis matrix. The simplest change of basis matrix is one which can be used to convert from some basis to the standard basis. This is obtained simply by writing down the coordinate vectors of the basis polynomials and putting them into a matrix. The process can also be reversed by using the inverse matrix.

Examples:

- Find the change of basis matrix which converts from the Vandermonde basis $\{(t-1)^2, (t-2)^2, (t-3)^2\}$ to the standard basis of P_2. Since $(t-1)^2 = 1 - 2t + t^2$, it has coordinate vector with respect to the standard basis: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Similarly, $(t-2)^2$ and $(t-3)^2$ have coordinate vectors: $\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 9 \\ -6 \\ 1 \end{pmatrix}$. The change of basis matrix is thus: $\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix}$. This can be used to convert the polynomial $3(t-1)^2 - 2(t-2)^2 + 4(t-3)^2$ to the standard basis as follows: Since the coordinate vector of this polynomial is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ we multiply to get:

$$\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 31 \\ -22 \\ 5 \end{pmatrix}$$

  which says that $3(t-1)^2 - 2(t-2)^2 + 4(t-3)^2 = 31 - 22t + 5t^2$. But this is no surprise, since we can also work this out by simply multiplying these binomials and adding terms. What is a little more subtle, is the fact that we can also reverse this process with an inverse matrix.

- In the previous example we apply the inverse matrix to the matrix equation and obtain:

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} & 3 \\ -1 & -2 & -3 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 31 \\ -22 \\ 5 \end{pmatrix}$$

  This gives the same information about the polynomials in reverse. It can also be used to find the coefficients for any polynomial in standard basis, converted to the Vandermonde basis. For example, suppose we want to convert the polynomial $2 - 3t + t^2$ into the Vandermonde basis from the previous example. This means we want to find coefficients so that:

$$2 - 3t + t^2 = a_0(t-1)^2 + a_1(t-2)^2 + a_2(t-3)^2.$$

  So we simply apply the inverse matrix to the stanard basis coordinate vector, to obtain:

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{4} & 3 \\ -1 & -2 & -3 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1 \\ -\frac{1}{4} \end{pmatrix}$$

  This is saying that $2 - 3t + t^2 = \frac{1}{4}(t-1)^2 + (t-2)^2 - \frac{1}{4}(t-3)^2$ which can also be checked by multiplying out the right side.

Bernstein polynomials and Bernstein basis

The Bernstein polynomials of degree d are labelled as:

$$B_0^d(t), B_1^d(t), B_2^d(t), \ldots, B_d^d(t).$$

Each one is defined as:

$$B_i^d(t) = \binom{d}{i}(1-t)^{d-i}t^i,$$

where the binomial coefficient $\binom{d}{i}$ is defined as:

$$\binom{d}{i} = \frac{d!}{(d-i)!i!}.$$

The binomial coefficients are better computed with Pascal's Identity:

$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i}$$

and the fact that $\binom{d}{0} = \binom{d}{d} = 1$. This identity is the basis for Pascal's Triangle, in which row d consists of $\binom{d}{0}\ \binom{d}{1} \cdots \binom{d}{d}$.

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\ \text{etc.} \end{array}$$

The binomial coefficients also count the number of subsets of size i in a set of size d.

Examples:

- The number of subsets of size two in the set $\{1,2,3,4\}$ is six, and sets are clearly $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$. This is the binomial coefficient $\binom{4}{2}$ which equals 6. It is also the middle element of the row

  $$1\ \ 4\ \ 6\ \ 4\ \ 1$$

  in Pascal's Triangle, and Pascal's Identity is:

  $$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6.$$

Bernstein basis

The set of Bernstein polynomials $\{B_0^d(t), B_1^d(t), B_2^d(t), \ldots, B_d^d(t)\}$ is a basis of P_d, called the **Bernstein basis** of P_d.

Examples:

- The Bernstein basis of P_1 is $\{B_0^1(t), B_1^1(t)\} = \{1-t, t\}$.
- The Bernstein basis of P_2 is $\{B_0^2(t), B_1^2(t), B_2^2(t)\} = \{(1-t)^2, 2(1-t)t, t^2\}$.
- The Bernstein basis of P_3 is $\{B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)\} = \{(1-t)^3, 3(1-t)^2t, 3(1-t)t^2, t^3\}$.

More Change of Basis

Find the change of basis matrix from $B(2) = \{B_0^2(t), B_1^2(t), B_2^2(t)\}$ to the top-down basis $B_1 = \{(t-1)^2, t-1, (t-2)^2\}$.

The simplest method is to first change from $B(2)$ to standard basis S_2 and then from S_2 to B_1. The change of basis from $B(2)$ to S_2 is obtained as above, by first expanding the polynomials:

$B_0^2(t) = (1-t)^2 = 1 - 2t + t^2$, with coordinate vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

$B_1^2(t) = 2(1-t)t = 2t - 2t^2$, with coordinate vector $\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, and

$B_2^2(t) = t^2$, with coordinate vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then the change of basis matrix which converts from $B(2)$ to S_2 is:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Next we need the change of basis matrix from S_2 to B_1. Again, we first convert the other way, from B_1 to S_2:

$(t-1)^2 = 1 - 2t + t^2$, has coordinate vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

$t - 1$, has coordinate vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and

$(t-2)^2 = 4 - 4t + t^2$, has coordinate vector $\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$.

So the change of basis matrix which converts from B_1 to S_2 is:

$$A_2 = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 1 & -4 \\ 1 & 0 & 1 \end{pmatrix}.$$

We need the inverse, which changes from S_2 to B_1:

$$A_2^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

The final matrix which changes from $B(2)$ to B_1 is then the product:

$$A_2^{-1}A_1 = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can test this on a particular polynomial, say $2(1-t)^2 + 3(2(1-t)t) + 4t^2$. This has coordinate vector $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ with respect to the basis $B(2)$. Using the matrix above, we can convert it to B_1:

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 4 \end{pmatrix},$$

which is saying that $2(1-t)^2 + 3(2(1-t)t) + 4t^2 = -4(t-1)^2 + 10(t-1) + 4(t-2)^2$.

This is confirmed by multiplying both polynomials out to the standard basis:

$$2(1-t)^2 + 3(2(1-t)t) + 4t^2 = 2t + 2,$$

$$-4(t-1)^2 + 10(t-1) + 4(t-2)^2 = 2t + 2.$$

In the exercises, we construct polynomials that pass through certain points, or have required derivatives at certain points. This is a special case of polynomial interpolation. Here we do a few examples with the method of linear systems.

Examples:

- Find the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ which satisfies: $p(0) = 2$, $p(1) = -3$, and $p(2) = 0$. To solve for the coefficients, we set up three linear equations:

$$\begin{array}{ccccccc} a_0 & + & a_1 \cdot 0 & + & a_2 \cdot 0^2 & = & 2 \\ a_0 & + & a_1 \cdot 1 & + & a_2 \cdot 1^2 & = & -3 \\ a_0 & + & a_1 \cdot 2 & + & a_2 \cdot 2^2 & = & 0 \end{array}$$

  This is equivalent to the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -3 \\ 1 & 2 & 4 & 0 \end{array}\right)$$

  The solution can be obainted by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -3 \\ 1 & 2 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 2 & 4 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 2 & 8 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

  which means that $a_0 = 2$, $a_1 = -9$, and $a_2 = 4$, and the polynomial is $p(t) = 2 - 9t + 4t^2$. Checking, we see that indeed $p(0) = 2$, $p(1) = -3$, and $p(2) = 0$.

- Find the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ which satisfies: $p(0) = 2$, $p'(0) = 1$, and $p(1) = 3$. To solve for the coefficients, we set up three linear equations, this time using both $p(t)$ and $p'(t) = a_1 + 2a_2 t$:

$$\begin{array}{ccccccc} a_0 & + & a_1 \cdot 0 & + & a_2 \cdot 0^2 & = & 2 \\ & & a_1 & + & 2 \cdot a_2 \cdot 0 & = & 1 \\ a_0 & + & a_1 \cdot 1 & + & a_2 \cdot 1^2 & = & 3 \end{array}$$

  This is equivalent to the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

The solution can be obainted by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

which means that $a_0 = 2$, $a_1 = 1$, and $a_2 = 0$, and the polynomial is $p(t) = 2 + t + 0 \cdot t^2 = 2 + t$. Checking, we see that indeed $p(0) = 2$, $p'(0) = 1$, and $p(1) = 3$. Note: this could have been predicted by the conditions, since heading out from $(0, 2)$ with a slope of 1 will take you directly to $(1, 3)$. We will see later, in the section on interpolation, that the solution to this type of problem is unique within the vector space P_d (in this case P_2) so in this case we know that there is no parabola satisfying the conditions.