Lecture 20

Main Points:

- Vector spaces of splines with various orders of continuity
- Knot sequences for spline bases

Vector spaces of splines with various orders of continuity

Recall that a standard basis for polynomial splines can be formed by including first a basis of the polynomials of degree at most d, and then including at each break point u_i $(1 \le i \le k-1)$ the shifted power functions $(t-u_i)_+^j$ where $r+1 \le j \le d$.

This gave us the basis for $P_{d,r}^k[u_0,\dots,u_k]$:

$$\{1, t, t^2, \dots, t^d, (t-u_1)^{r+1}, \dots, (t-u_1)^d, \dots$$
$$\dots (t-u_{k-1})^{r+1}, \dots, (t-u_{k-1})^d\}.$$

The same can be done if r is allowed to vary. In particular, we can give a vector of continuity conditions $\mathbf{r} = (r_1,\dots,r_{k-1})$ and define the vector space of splines with continuity of order at least r_i at each u_i, $1 \le i \le k-1$. We call this vector space:

$$P_{d,\mathbf{r}}^k[u_0,\dots,u_k],$$

where $\mathbf{r} = (r_1,\dots,r_{k-1})$.

In summary, we define $P_{d,\mathbf{r}}^k[u_0,\dots,u_k]$, with $\mathbf{r} = (r_1,\dots,r_{k-1})$ to be the subset of $P_d^k[u_0,\dots,u_k]$ consisting of functions which have r_i continuous derivatives at the break point u_i, for $i = 1,\dots,k-1$. This is equivalent to requiring that $p_i^{(j)}(u_i) = p_{i+1}^{(j)}(u_i)$ for $j = 0,\dots,r_i$ and $i = 1,\dots,k-1$. In other words, the polynomials must match in their function values (zeroth derivative) and their first r_i derivatives at each break point u_i, for $i = 1,\dots,k-1$.

A basis for $P_{d,\mathbf{r}}^k[u_0,\dots,u_k]$, with $\mathbf{r} = (r_1,\dots,r_{k-1})$ is:

$$\{1, t, t^2, \dots, t^d, (t-u_1)^{r_1+1}, \dots, (t-u_1)^d, \dots$$
$$\dots (t-u_{k-1})^{r_{k-1}+1}, \dots, (t-u_{k-1})^d\}.$$

Examples:

- A basis for $P_{2,\mathbf{r}}^4[0,1,2,3,4]$ with $\mathbf{r} = (1,0,-1)$ is:

$$\{1, t, t^2, (t-1)_+^2, (t-2)_+^1, (t-2)_+^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2\}.$$

- A basis for $P_{3,\mathbf{r}}^4[0,1,2,3,4]$ with $\mathbf{r} = (1,2,0)$ is:

$$\{1, t, t^2, t^3, (t-1)_+^2, (t-1)_+^3, (t-2)_+^3, (t-3)_+^1, (t-3)_+^2, (t-3)_+^3\}.$$

Knot Sequences:

A knot sequence $\mathbf{t}$ is defined as a non-decreasing sequence of real numbers $\{t_0, t_1, \dots, t_N\}$, ie. $t_0 \le t_1 \le \dots \le t_N$. The multipicity of $t_i = c$ in the knot sequence is the greatest number of consecutive values in the knot sequence equal to c.

$sp_d(\mathbf{t})$ is a set of shifted power functions determined by the knot sequence $\mathbf{t}$ so that whenever $t_i = t_{i+1} = \cdots = t_{i+q}$ then we have $(t-t_i)_+^d, (t-t_i)_+^{d-1}, \ldots, (t-t_i)_+^{d-q}$ is part of $sp_d(\mathbf{t})$. If the multiplicities are all equal to 1, then $sp_d(\mathbf{t})$ is just the sequence of shifted power functions $(t-t_i)_+^d$, $i = 0, \ldots, d$.

Examples:

- The knot sequence $\mathbf{t} = \{0, 1, 1, 2, 3, 3, 3\}$ can be used to form the sets of shifted power functions:

  $$sp_2(\mathbf{t}) = \{(t-0)_+^2, (t-1)_+^2, (t-1)_+^1, \{(t-2)_+^2, (t-3)_+^2, (t-3)_+^1, (t-3)_+^0\},$$

  and

  $$sp_3(\mathbf{t}) = \{(t-0)_+^3, (t-1)_+^3, (t-1)_+^2, \{(t-2)_+^3, (t-3)_+^3, (t-3)_+^2, (t-3)_+^1\}.$$

  We can also create other sets of functions for higher d.

- The knot sequence $\mathbf{t} = \{1, 1, 2\}$ can be used to form the set of shifted power functions:

  $$sp_2(\mathbf{t}) = \{(t-1)_+^2, (t-1)_+^1, (t-2)_+^2\}.$$

  This set of functions looks a lot like a top-down basis of P_2. But these functions are not polynomials. They are in fact piecewise polynomial functions. However, if we restrict them to an interval such as $[3, 7]$, then they become simply:

  $$\{(t-1)^2, t-1, (t-2)^2\},$$

  which is indeed a top-down basis of P_2 restricted to the interval $[3, 7]$. We will make use of this observation in describing bases for spline vector spaces.

- A basis for $P_2^3[1, 2, 3, 4]$ can be given as:

  $$sp_2(\mathbf{t}) = \{(t-1)_+^2, (t-1)_+^1, (t-1)_+^0, (t-2)_+^2, (t-2)_+^1, (t-2)_+^0, (t-3)_+^2, (t-3)_+^1, (t-3)_+^0\},$$

  with $\mathbf{t} = \{1, 1, 1, 2, 2, 2, 3, 3, 3\}$. Note: We have written the terms with decreasing orders. This is convenient since the default case, for knot sequences, is to start with the highest power and only list lower powers if there are multiple copies of the same number. In the correspondence to triples in P_2^3, this basis corresponds to:

  $$\{(1, 1, 1), (t-1, t-1, t-1), ((t-1)^2, (t-1)^2, (t-1)^2),$$

  $$(0, 1, 1), (0, t-2, t-2), (0, (t-2)^2, (t-2)^2),$$

  $$(0, 0, 1), (0, 0, t-3), (0, 0, (t-3)^2)\}.$$

  Note: these terms are written in the usual degree increasing order, as we have done before for k-tuples. (Can you see which triple corresponds to which function?)

- A basis for $P_{2,\mathbf{r}}^3[1, 2, 3, 4]$, with $\mathbf{r} = (1, 0)$, can be given as:

  $$sp_2(\mathbf{t}) = \{(t-0)_+^2, (t-0)_+^1, (t-0)_+^0, (t-2)_+^2, (t-3)_+^2, (t-3)_+^1\},$$

  with $\mathbf{t} = \{0, 0, 0, 2, 3, 3\}$. Note that on the interval $[1, 4]$ the functions $(t-0)_+^2, (t-0)_+^1$ and $(t-0)_+^0$ are exactly the same as the standard basis t^2, t and 1.

Multiplicity Vector

Related to the continuity vector $\mathbf{r} = (r_1, \ldots, r_{k-1})$ is the multiplicity vector:

$$\mathbf{m} = (m-1, \ldots, m_{k-1}).$$

Each m_i is defined to be:

$$m_i = d - r_i,$$

which also coincides with the number of functions in the list:

$$(t-u_i)_+^{r_i+1}, \ldots, (t-u_i)_+^d.$$

Also, since the orders of continuity satisfy $-1 \leq r_i \leq d-1$, we have:

$$1 \leq m_i \leq d+1.$$

Dimension of spline vector spaces with different orders of continuity at each break point

The dimension of $P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$ is given as:

$$d+1+\sum_{i=1}^{k-1}(d-r_i) = d+1+\sum_{i=1}^{k-1} m_i.$$

Note: This basis consists of a basis of P_d (the standard basis) and also $d-r_i$ shifted power functions at u_i, for $i = 1, \ldots, k-1$.

General shifted power bases

The vector space $V = P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$, with $\mathbf{r} = (r_1, \ldots, r_{k-1})$ has many different shifted power bases, which can be given by specifying different knot sequences

$$\mathbf{t} = \{t_0, t_1, \ldots, t_N\},$$

where

$$N+1 = d+1+\sum_{i=1}^{k-1} m_i$$

is the dimension of V. The first $d+1$ values in the knot sequence must correspond to a basis of polynomials which replaces the standard basis $\{1, t, t^2, \ldots, t^d\}$. We can construct such a set by choosing shifted power functions $(t-c)_+^j$ whose value c is less than or equal to u_0. This way, any such function is simply a polynomial on the interval $[u_0, u_k]$. If we choose

$$t_0 \leq t_1 \leq \cdots \leq t_d \leq u_0,$$

then we find that the set of shifted power functions associated to this set

$$sp_d(\{t_0, t_1, \ldots, t_d\})$$

is in fact a top down basis of P_d, when restricted to the interval $[u_0, u_k]$. Note: the right half of the sequence:

$$\{t_{d+1}, t_{d+2}, \ldots, t_d\} = \{u_1, \ldots, u_1, \ldots, u_{k-1}, \ldots, u_{k-1}\}$$

is made up of the break points with their multiplicities given by the m_i.

Examples: The vector space $P_{2,\mathbf{r}}^3[1,2,3,4]$, with $\mathbf{r} = (1,0)$, has multiplicity vector $\mathbf{m} = (1,2)$, and has the bases $sp_2(\mathbf{t})$ for all of the following knot sequences:

- $\mathbf{t} = \{1,1,1,2,3,3\}$, with basis:

  $$\{(t-1)_+^2, (t-1)_+^1, (t-1)_+^0, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a shifted basis of polynomials.)

- $\mathbf{t} = \{-3,-2,-1,2,3,3\}$, with basis:

  $$\{(t+3)_+^2, (t+2)_+^2, (t+1)_+^2, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a Vandermonde basis of polynomials.)

- $\mathbf{t} = \{-2, -2, -1, 2, 3, 3\}$, with basis:

  $$\{(t+2)_+^2, (t+2)_+^1, (t+1)_+^2, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a top-down basis of polynomials.)

Definition of B-Splines with divided differences

The definition of B-splines with the divided difference formulation: Given a knot sequence $\mathbf{t}$ with $t_0 \leq t_1 \leq \ldots \leq t_N$, we define for all $d \geq 0$ and for $0 \leq i \leq N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d.$$

Note: The divided difference is computed with t as constant and x as the variable. We can think of $g(x) = (t-x)_+^d$ as playing the role of the data function.

Examples:

- We will compute the piecewise polynomial form for $\mathcal{B}_i^0(t)$. First, we write:

  $$\mathcal{B}_i^0(t) = (-1)(t_{i+1} - t_i)[t_i, t_{i+1}]((t-x)_+^0.$$

  Now suppose that

  $$t_i < t < t_{i+1}.$$

  Then the data function $g(x)$ has values:

  $$g(t_i) = 1, \qquad \text{and} \qquad g(t_{i+1}) = 0.$$

  Now recall that the divided difference $[t_i, t_{i+1}]g$ is equal to the coefficient of x in the interpolating polynomial $p(x)$ which matches g at the values $x = t_i$ and $x = t_{i+1}$. But this is just the slope of the line through the two points $(t_i, 1)$ and $(t_{i+1}, 0)$ which we can compute as

  $$[t_i, t_{i+1}]((t-x)_+^0 = \frac{-1}{t_{i+1} - t_i}.$$

  So, when $t_i < t < t_{i+1}$, then

  $$\begin{aligned}
  \mathcal{B}_i^0(t) &= (-1)(t_{i+1} - t_i)[t_i, t_{i+1}]((t-x)_+^0 \\
  &= (-1)(t_{i+1} - t_i)\frac{-1}{t_{i+1} - t_i} \\
  &= 1
  \end{aligned}$$