Lecture 22

Main Points:

- B-spline Recursion formula
- Sums of shifted power functions

B-splines of degree d for a knot sequence

Given a knot sequence $\mathbf{t}$ with $t_0 \leq t_1 \leq \ldots \leq t_N$, we define for any degree $d \geq 0$, and for $0 \leq i \leq N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d$$

The set $\mathcal{B}_d(\mathbf{t})$ or $\mathcal{B}_{d,\mathbf{t}}$ is the set of B-splines associated to the knot sequence $\mathbf{t}$: $\mathcal{B}_d(\mathbf{t}) = \{\mathcal{B}_0^d(t), \ldots, \mathcal{B}_{N-d-1}^d(t)\}$

Examples:

- We will compute the piecewise polynomial form for $\mathcal{B}_i^1(t)$. First, we write:

$$\mathcal{B}_i^1(t) = (-1)^2(t_{i+2} - t_i)[t_i, t_{i+1}, t_{i+2}](t-x)_+^1.$$

This definition uses the continuous shifted power function

$$g(x) = (t-x)_+^1 = \begin{cases} t-x, & x \leq t \\ 0, & x > t \end{cases}$$

If $t < t_i$ then we have $g(t_i) = g(t_{i+1}) = g(t_{i+2}) = 0$ and thus the divided difference

$$[t_i, t_{i+1}, t_{i+2}](t-x)_+^1 = 0, \quad \text{for} \ \ t < t_i.$$

Also, if $t \geq t_{i+2}$ then $g(x)$ takes the values of the straight line $t-x$ at each of these inputs, which means that the interpolating polynomial which matches g at those values is in fact a line. But then the coefficient of x^2 in this polynomial must be zero. So again we have:

$$[t_i, t_{i+1}, t_{i+2}](t-x)_+^1 = 0, \quad \text{for} \ \ t \geq t_{i+2}.$$

Now suppose that

$$t_i \leq t < t_{i+1}.$$

Then we compute $[t_i, t_{i+1}, t_{i+2}]g$ from the divided difference table:

$$\begin{array}{cccc}
t_i & t-t_i & & \\
 & & -\frac{t-t_i}{t_{i+1}-t_i} & \\
t_{i+1} & 0 & & \frac{t-t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = [t_i, t_{i+1}, t_{i+2}]g \\
 & & 0 & \\
t_{i+2} & 0 & &
\end{array}$$

Then we have:

$$\mathcal{B}_i^1(t) = (t_{i+2} - t_i)\frac{t-t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = \frac{t-t_i}{t_{i+1}-t_i}$$

for $t_i \leq t < t_{i+1}$. Note that this is a line segment which starts at $(t_i, 0)$ and ends by approaching the point $(t_{i+1}, 1)$.

Now suppose that

$$t_{i+1} \leq t \leq t_{i+2}.$$

Again we compute $[t_i, t_{i+1}, t_{i+2}]g$ from the divided difference table:

$$\begin{array}{llll}
t_i & t - t_i & & \\
 & & -1 & \\
t_{i+1} & t - t_{i+1} & & \frac{1}{t_{i+2}-t_i}\left(1 - \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}\right) = [t_i, t_{i+1}, t_{i+2}]g \\
 & & -\frac{t-t_{i+1}}{t_{i+2}-t_{i+1}} & \\
t_{i+2} & 0 & &
\end{array}$$

Then we have:

$$\mathcal{B}_i^1(t) = (t_{i+2} - t_i)\frac{1}{t_{i+2} - t_i}\left(1 - \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}}\right) = 1 - \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}}$$

for $t_{i+1} \le t \le t_{i+2}$. Note that this is a line segment which starts at $(t_{i+1}, 1)$ and ends at the point $(t_{i+2}, 0)$.

We can now put the whole function together, to get:

$$\mathcal{B}_i^1(t) = \begin{cases} 0, & t < t_i \\ \frac{t-t_i}{t_{i+1}-t_i}, & t_i \le t < t_{i+1} \\ 1 - \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}, & t_{i+1} \le t \le t_{i+2} \\ 0, & t > t_{i+2} \end{cases}$$

The graph of this B-spline is a 'hat function' which is zero outside and at the endpoints of the interval $[t_i, t_{i+2}]$, and reaches the value 1 at the value t_{i+1} in between.

The B-spline recursion formula

The (DeBoor-Cox) recursion formula for B-splines of degree d associated to a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$ is:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i}\mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}}\mathcal{B}_{i+1}^{d-1}(t)$$

The base case is:

$$\mathcal{B}_i^0(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & elsewhere \end{cases}$$

Special case: if $t_{i+d+1} = t_i$ then the B-spline $\mathcal{B}_i^d(t)$ is defined to be zero.

Note: The base case and special case above are necessary if this is taken to be the definition of B-splines, which it is in many texts. However, the recursion follows directly from the definition using the divided differences. We have already worked out the base case $d = 0$ with the definition. Next we will prove the recursion using the same definition.

Proof of the B-spline recursion formula

First, we write the function $(t - x)_+^d$ as a product:

$$(t - x)_+^d = g(x)h(x) = (t - x)(t - x)_+^{d-1}.$$

We will use the Leibniz formula for divided difference of a product applied to the divided difference in the B-spline definition:

$$[t_i, t_{i+1}, \ldots, t_{i+d+1}](t - x)_+^d = [t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h.$$

Recalling the Leibniz formula, we have:

$$[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h = [t_i]g \cdot [t_i, \ldots, t_{i+d+1}]h + [t_i, t_{i+1}]g \cdot [t_{i+1}, \ldots, t_{i+d+1}]h + \cdots$$

where the terms described by the three dots will shortly be seen to be all equal to zero. First we describe the two terms above. In the first term we have

$$[t_i]g = g(t_i) = t - t_i$$

from the definition of the function $g(x) = t - x$. In the second term we have

$$[t_i, t_{i+1}]g = \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} = \frac{t - t_{i+1} - (t - t_i)}{t_{i+1} - t_i} = \frac{t_i - t_{i+1}}{t_{i+1} - t_i} = -1.$$

This can also be deduced directly from the definition of the divided difference as the coefficient of x in the interpolating polynomial since in this case that polynomial must simply be the line $t - x$ with slope -1.

In order to see that the higher terms are all zero, we use the same interpretation of the divided difference for $[t_i, t_{i+1}, t_{i+2}]g$. In this case, it is equal to the coefficient of x^2 in the interpolating polynomial which matches g at those three values. But g is still a linear polynomial $t - x$, so $p(x) = t - x$, and thus the coefficient of x^2 is zero. This holds for all higher terms as well.

So far we have:

$$[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h = (t - t_i) \cdot [t_i, \ldots, t_{i+d+1}]h + (-1) \cdot [t_{i+1}, \ldots, t_{i+d+1}]h.$$

Next, we apply the recursive form of the divided difference to $[t_i, \ldots, t_{i+d+1}]h$, and simplify:

$$\begin{aligned}
[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h &= (t - t_i) \cdot [t_i, \ldots, t_{i+d+1}]h + (-1) \cdot [t_{i+1}, \ldots, t_{i+d+1}]h \\
&= (t - t_i) \left(\frac{[t_{i+1}, \ldots, t_{i+d+1}]h - [t_i, \ldots, t_{i+d}]h}{t_{i+d+1} - t_i} \right) - [t_{i+1}, \ldots, t_{i+d+1}]h \\
&= \left(\frac{t - t_i}{t_{i+d+1} - t_i} - 1 \right) [t_{i+1}, \ldots, t_{i+d+1}]h - \frac{(t - t_i)}{t_{i+d+1} - t_i} [t_i, \ldots, t_{i+d}]h \\
&= \frac{t - t_{i+d+1}}{t_{i+d+1} - t_i} [t_{i+1}, \ldots, t_{i+d+1}]h - \frac{(t - t_i)}{t_{i+d+1} - t_i} [t_i, \ldots, t_{i+d}]h
\end{aligned}$$

Next, recall the definitions of two lower degree B-splines:

$$\mathcal{B}_i^{d-1}(t) = (-1)^d (t_{i+d} - t_i) [t_i, t_{i+1}, \ldots, t_{i+d}](t - x)_+^{d-1}$$

and

$$\mathcal{B}_{i+1}^{d-1}(t) = (-1)^d (t_{i+d+1} - t_{i+1}) [t_{i+1}, t_{i+2}, \ldots, t_{i+d+1}](t - x)_+^{d-1}.$$

These allow us to solve for the divided differences in the above formula:

$$[t_i, \ldots, t_{i+d}]h = \frac{(-1)^d}{(t_{i+d} - t_i)} \mathcal{B}_i^{d-1}(t),$$

and

$$[t_{i+1},\ldots,t_{i+d+1}]h = \frac{(-1)^d}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t).$$

Putting all of this together, we then have:

$$\begin{aligned}
\mathcal{B}_i^d(t) &= (-1)^{d+1}(t_{i+d+1}-t_i)[t_i,t_{i+1},\ldots,t_{i+d+1}]g\cdot h \\
&= (-1)^{d+1}(t_{i+d+1}-t_i)\left(\frac{t-t_{i+d+1}}{t_{i+d+1}-t_i}[t_{i+1},\ldots,t_{i+d+1}]h - \frac{(t-t_i)}{t_{i+d+1}-t_i}[t_i,\ldots,t_{i+d}]h\right) \\
&= (-1)^{d+1}\left((t-t_{i+d+1})[t_{i+1},\ldots,t_{i+d+1}]h-(t-t_i)[t_i,\ldots,t_{i+d}]h\right) \\
&= (-1)^{d+1}\left((t-t_{i+d+1})\frac{(-1)^d}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t)-(t-t_i)\frac{(-1)^d}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t)\right) \\
&= (-1)^{2d+1}\left(\frac{(t-t_{i+d+1})}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t)-\frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t)\right) \\
&= -\frac{(t-t_{i+d+1})}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t)+\frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t) \\
&= \frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t)+\frac{(t_{i+d+1}-t)}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t)
\end{aligned}$$

This establishes the Recursion formula for B-splines.

Examples:

- Let $\mathbf{t}=\{0,1,2,3,4,5\}$. Then we can write the degree 1 B-spline $\mathcal{B}_0^1(t)$ in terms of degree 0 B-splines according to the recursion formula: (using the first three knot values 0, 1, 2)

  $$\mathcal{B}_0^1(t)=\frac{t-0}{1-0}\mathcal{B}_0^0(t)+\frac{2-t}{2-1}\mathcal{B}_1^0(t)=t\mathcal{B}_0^0(t)+(2-t)\mathcal{B}_1^0(t).$$

  We can also obtain the piecewise form from this recursive form:

  $$\begin{aligned}
  \mathcal{B}_0^1(t) &= t\mathcal{B}_0^0(t)+(2-t)\mathcal{B}_1^0(t) \\
  &= t\begin{cases}0, & t<0\\ 1, & 0\le t<1\\ 0, & t\ge 1\end{cases} + (2-t)\begin{cases}0, & t<1\\ 1, & 1\le t<2\\ 0, & t\ge 2\end{cases}
  \end{aligned}$$

$$= t\begin{cases} 0, & t<0 \\ 1, & 0\le t<1 \\ 0, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + (2-t)\begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ 1, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

$$= \begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 0, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + \begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

$$= \begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

- By a similar calculation, or by simply shifting $\mathcal{B}_0^1(t)$ to the right one unit, we can obtain:

$$\mathcal{B}_1^1(t) = \begin{cases} 0, & t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases}$$

- With the same knot sequence $\mathbf{t} = \{0,1,2,3,4,5\}$, we can write the degree 2 B-spline $\mathcal{B}_0^2(t)$ in terms of degree 1 B-splines according to the recursion formula: (using the first four knot values 0, 1, 2, 3)

$$\mathcal{B}_0^2(t) = \frac{t-0}{3-1}\mathcal{B}_0^1(t) + \frac{3-t}{2-0}\mathcal{B}_1^1(t) = \frac{1}{2}t\mathcal{B}_0^1(t) + \frac{1}{2}(3-t)\mathcal{B}_1^1(t).$$

We can also obtain the piecewise form from this recursive form:

$$\begin{aligned} \mathcal{B}_0^2(t) &= \frac{1}{2}t\mathcal{B}_0^1(t) + \frac{1}{2}(3-t)\mathcal{B}_1^1(t) \\ &= \frac{1}{2}t\begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + \frac{1}{2}(3-t)\begin{cases} 0, & t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} \\ &= \frac{1}{2}t\begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} + \frac{1}{2}(3-t)\begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} \\ &= \begin{cases} 0, & t<0 \\ \frac{1}{2}t^2, & 0\le t<1 \\ \frac{1}{2}t(2-t), & 1\le t<2 \\ 0, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} + \begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ \frac{1}{2}(3-t)(t-1), & 1\le t<2 \\ \frac{1}{2}(3-t)^2, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} \end{aligned}$$

$$
= \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 \leq t < 1 \\ \frac{1}{2}t(2-t) + \frac{1}{2}(3-t)(t-1), & 1 \leq t < 2 \\ \frac{1}{2}(3-t)^2, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}
$$

$$
= \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 \leq t < 1 \\ -t^2 + 3t - \frac{3}{2}, & 1 \leq t < 2 \\ \frac{1}{2}(3-t)^2, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}
$$